

# Quaternionic Quantization Principle in General Relativity and Supergravity

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A generalized quantization principle is considered, which incorporates nontrivial commutation relations of the components of the variables of the quantized theory with the components of the corresponding canonical conjugated momenta referring to other space-time directions. The corresponding commutation relations are formulated by using quaternions. At the beginning, this extended quantization concept is applied to the variables of quantum mechanics. The resulting Dirac equation and the corresponding generalized expression for plane waves are formulated and some consequences for quantum field theory are considered. Later, the quaternionic quantization principle is transferred to canonical quantum gravity. Within quantum geometrodynamics as well as the Ashtekar formalism the generalized algebraic properties of the operators describing the gravitational observables and the corresponding quantum constraints implied by the generalized representations of these operators are determined. The generalized algebra also induces commutation relations of the several components of the quantized variables with each other. Finally, the quaternionic quantization procedure is also transferred to  $\mathcal{N} = 1$  supergravity. Accordingly, the quantization principle has to be generalized to be compatible with Dirac brackets, which appear in canonical quantum supergravity.

## I. INTRODUCTION

The unification of quantum theory with general relativity is probably the most important research topic in contemporary fundamental theoretical physics. Various approaches exist to obtain a quantum theory of gravity. In the existing literature one can distinguish between two classes of theories. One class of theories presupposes usual general relativity, perhaps in a modified formulation, and then transfers the quantization principle of quantum theory to the corresponding degrees of freedom contained in the gravitational field. This is performed by canonical quantization or covariant quantization. The other class of theories assumes a modification of usual general relativity by presupposing an extended geometrical structure of space-time or an extended dynamics of the gravitational field for example, and then uses the general quantization principle of quantum theory as well. This means that the quantization principle and thus quantum theory remains usually completely unchanged.

But in principle it is also thinkable that quantum theory, this means the quantization principle, which determines the properties of the corresponding quantized theory, if a classical theory is presupposed, has to be generalized instead of the theory, which shall be formulated quantum theoretically. This means that it is not only possible to consider the presupposed geometry or dynamics of usual general relativity as an approximation to a more general gravity theory, but also to consider the quantization principle usually used in quantum mechanics, quantum field theory and approaches to a quantum description of general relativity as approximation to a more general quantization principle.

The generalized uncertainty principle in quantum mechanics as well as noncommutative geometry represent extensions of the quantum properties of the variables of quantum mechanics. The generalized uncertainty principle, developed in [1],[2],[3],[4],[5],[6],[7], postulates generalized commutation relations between the position operators and the corresponding momentum operators. Noncommutative geometry, originally considered in [8], postulates besides the commutation relations between the position operators and the corresponding momentum operators also commutation relations between the several components of the position operator and this idea can be transferred to additional commutation relations between the several momenta, presupposed in [9] for example. A generalized uncertainty principle can, depending on the special scenario, also imply commutation relations of the several components of the position operator with each other and this holds analogously for the momentum operator. These concepts can be interpreted as fundamental properties of nature and thus they would belong to quantum theory itself and accordingly represent a generalization of the concept of quantization. If this is postulated, then these generalized quantization principles have also to be transferred to the quantization of general relativity and thus the gravitational field what differs from the formulation of classical general relativity on noncommutative space-time [10] or even usual quantum general relativity on noncommutative space-time [11],[12],[13],[14]. In [15],[16],[17],[18],[19] ideas to transfer the concept of a generalized uncertainty principle to gravity can be found, but in [20] and [21] the generalized uncertainty principle has really been transferred to the variables of canonical quantum gravity and quantum cosmology,

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whereas in [22] the concept of noncommutative geometry has been transferred to the components of the tetrad field. An extension of the field theoretic quantization principle to a nonlocal quantization principle has been considered in [23].

In the present paper is suggested an approach to generalize the quantization principle of quantum theory, which seems concerning its application in quantum mechanics as a natural extension of the concept of noncommutative geometry. In noncommutative geometry the commutation relations of quantum mechanics are extended by commutation relations between the several components of the position operator. In the quantization concept presented in this paper are not only postulated nontrivial commutation relations between the components of the variables of the theory, which has to be quantized, and the corresponding components of the canonical conjugated variables belonging to the same space-time direction, but also nontrivial commutation relations with the components of the canonical conjugated variables belonging to other space-time directions. This generalized quantization principle of general quantum theory is formulated based on the mathematical concept of quaternions. Concretely, the additional commutation relations are assumed to be of the same shape, but are not proportional to the imaginary unit, but to another direction in the space of quaternions. If all these commutation relations were assumed to be proportional to the usual imaginary unit of the space of complex numbers, the tensor defining the quantization would not be invertible and in case of quantum mechanics no plane waves could be defined.

The generalized quantization principle, which could be called as quaternionic quantization principle, is first studied in the simplest case of quantum mechanics and after this it is transferred to general relativity and  $\mathcal{N} = 1$  supergravity. Of course, this kind of generalization of the quantization principle presupposes canonical quantization and accordingly the canonical formulations of general relativity and  $\mathcal{N} = 1$  supergravity have to be considered.

Quaternions with respect to physical theories have been studied in many areas of theoretical physics, for example quaternionic formulations have already been studied in the context of quantum mechanics, [24],[25],[26],[27],[28],[29],[30],[31],[32],[33],[34],[35],[36],[37],[38],[39],[40],[41] and quantum field theory [42],[43],[44],[45]. Concerning general relativity and geometry, quaternionic structures have been studied in [46],[47],[48],[49],[50],[51],[52],[53],[54],[55],[56],[57],[58],[59],[60],[61],[62],[63],[64],[65],[66],[67],[68],[69],[70],[71],[72]. Besides, quaternions have been used with respect to reformulations and extensions of particle physics and the standard model, [73],[74],[75],[76],[77],[78],[79],[80],[81],[82],[83],[84],[85], and especially with respect to supersymmetry as well as supergravity, [86],[87],[88],[89],[90],[91],[92],[93],[94],[95],[96]. But it is very important to mention that the quaternionic generalization of the quantization principle considered in the present paper differs decisively from earlier considerations, since here a completely new quantization principle as general physical concept is considered, whereas in the mentioned explorations a quaternionic reformulation of some physical theories or special geometrical scenarios through the introduction of quaternionic quantities have been considered. This means that the concept of quaternions serves as a mathematical concept to generalize the quantization principle as a physical concept, which becomes manifest with respect to quantum mechanics, quantum field theory as well as the quantization of general relativity and supergravity.

The paper is structured as follows: At the beginning is given a short introduction to the concept of quaternions. Then the suggested generalized quantization principle, which is based on quaternions and consists in the addition of nontrivial commutation relations between the components of the variables and the components of the corresponding canonical conjugated variables belonging to other space-time directions, is formulated for quantum mechanics as special manifestation. The corresponding free Dirac equation and the generalized plane waves are determined. After this the corresponding generalized propagator of a scalar field as well as the generalized gauge principle of electrodynamics related to local phase invariance are considered. Subsequently, the main aim of this paper is treated, the generalization of the canonical quantum description of general relativity by the idea of the quaternionic quantization principle. Accordingly the generalized commutation relations between the position and momentum operator are transferred to the variables of quantum geometrodynamics as well as to the variables of the Ashtekar formalism. Based on this, the corresponding quantum constraints are derived, especially the generalized Wheeler-DeWitt equation. As most intricate manifestation, the quaternionic quantization principle is also applied to an extension of classical general relativity, canonical supergravity namely and especially  $\mathcal{N} = 1$  supergravity. Since in the quantization procedure of supergravity appear Dirac brackets because of the second class constraints, the quaternionic quantization principle has to be generalized to be applicable to theories, which are usually quantized by defining Dirac brackets. This leads to much more complicated commutation relations. After this, the corresponding constraints are generalized and finally the inner product of canonical quantum supergravity has also to be reformulated.

## II. QUATERNIONS

In this section is given a short repetition of the concept of quaternions, which serves also to introduce the notation, which is used to express the quaternions. Quaternions are a generalization of complex numbers with two additional dimensions besides the usual real dimension and the usual imaginary dimension. This means that quaternions represent

elements of a four-dimensional vector space as number space. A quaternion can be represented in the following way:

$$q = a + bi + cj + dk, \quad (1)$$

where  $a, b, c$  and  $d$  are real numbers and  $i, j, k$  are quantities fulfilling the following relations:

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \quad i^2 = j^2 = k^2 = -1. \quad (2)$$

The corresponding conjugated quantity to a quaternion  $q$  defined in (1),  $q^*$ , is defined as

$$q^* = a - bi - cj - dk. \quad (3)$$

Of course, the space of the complex numbers represents a subspace of the space of the quaternions, which is built by all quaternions with  $c = d = 0$ . The norm of a quaternion denoted by  $|q|$  is given in analogy to the norm of a complex number by

$$|q| = \sqrt{q^*q} = \sqrt{a^2 + b^2 + c^2 + d^2}. \quad (4)$$

The quaternions can be represented by using the Pauli matrices, if  $1, i, j$  and  $k$  are related to the Pauli matrices the unity matrix in two dimensions included in the following way,

$$\mathbf{1} = \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = i_{\mathbb{C}}\sigma^3 = \begin{pmatrix} i_{\mathbb{C}} & 0 \\ 0 & -i_{\mathbb{C}} \end{pmatrix}, \quad j = i_{\mathbb{C}}\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = i_{\mathbb{C}}\sigma^1 = \begin{pmatrix} 0 & i_{\mathbb{C}} \\ i_{\mathbb{C}} & 0 \end{pmatrix}, \quad (5)$$

where  $i_{\mathbb{C}}$  denotes the usual complex unit. This is helpful to determine the inverse matrix of a quaternionic matrix for example, which can be represented by a complex matrix in this way.

### III. QUATERNIONIC QUANTIZATION IN QUANTUM MECHANICS

In this section the generalized quantization concept based on the mathematical concept of quaternions shall be introduced and this is done in the realm of quantum mechanics. The basic postulate of quantum mechanics consists in the fundamental commutation relation between the position and the corresponding momentum operators,

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta_\nu^\mu, \quad (6)$$

which defines their mathematical properties and constitutes the corresponding complex vector space of the possible states. The Planckian constant  $\hbar$  as well as the speed of light  $c$  are set equal to one throughout the paper,  $\hbar = c = 1$ . Greek indices refer to all coordinates of space-time, whereas Latin indices refer to the spatial coordinates of space-time in this paper. In usual quantum mechanics the components of the position operator fulfil only nontrivial commutation relations with the corresponding components of the momentum operator, which refer to the same space-time direction. Noncommutative geometry extends these relations by additional commutation relations between the several components of the space-time coordinates,  $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$ . As already mentioned in the explanation of the introduction, it suggests itself to generalize the quantization postulate (6) to a generalized quantization postulate, containing also nontrivial commutation relations between the components of the position operator and the components of the momentum operator not referring to the same space-time direction. If these commutation relations would also be postulated to be proportional to the complex imaginary unit  $i_{\mathbb{C}}$ , this would lead to a matrix, which is not invertible and thus no generalized plane waves as solutions for the free field equations could be defined. Therefore quaternions are introduced to enable the postulation of commutation relations being proportional to another direction in the space of the quaternions. Accordingly, a transition to the following fundamental commutation relation is suggested as generalization of usual quantum mechanics:

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta_\nu^\mu \quad \longrightarrow \quad [\hat{x}^\mu, \hat{p}_\nu] = \alpha_\nu^\mu, \quad (7)$$

where  $\alpha^\mu_\nu$  is a quaternionic tensor of second order which looks as follows:

$$\alpha^\mu_\nu = \begin{pmatrix} i & \varkappa j & \varkappa j & \varkappa j \\ \varkappa j & i & \varkappa j & \varkappa j \\ \varkappa j & \varkappa j & i & \varkappa j \\ \varkappa j & \varkappa j & \varkappa j & i \end{pmatrix}. \quad (8)$$

$i$  and  $j$  denote the units of the quaternionic number space defined in (2) and  $\varkappa$  denotes a dimensionless parameter defining the relation between the influence of the usual commutation relations and the additional ones. If  $\varkappa$  goes to zero, one obtains usual quantum mechanics as approximation to the theory presented in this paper. To generalize plane waves according to (7), the inverse matrix of  $\alpha^\mu_\nu$  will become important, which is given by

$$(\alpha^{-1})^\mu_\nu = \begin{pmatrix} a_i i + a_j j & b_i i + b_j j & b_i i + b_j j & b_i i + b_j j \\ b_i i + b_j j & a_i i + a_j j & b_i i + b_j j & b_i i + b_j j \\ b_i i + b_j j & b_i i + b_j j & a_i i + a_j j & b_i i + b_j j \\ b_i i + b_j j & b_i i + b_j j & b_i i + b_j j & a_i i + a_j j \end{pmatrix}, \quad (9)$$

where the coefficients  $a_i$ ,  $a_j$ ,  $b_i$  and  $b_j$  are defined as

$$a_i = -\frac{7\varkappa^2 + 1}{9\varkappa^4 + 10\varkappa^2 + 1}, \quad a_j = \frac{6\varkappa^3}{9\varkappa^4 + 10\varkappa^2 + 1}, \quad b_i = \frac{2\varkappa^2}{9\varkappa^4 + 10\varkappa^2 + 1}, \quad b_j = -\frac{3\varkappa^3 + \varkappa}{9\varkappa^4 + 10\varkappa^2 + 1}. \quad (10)$$

The quantization postulate (7) also implies nontrivial commutation relations between the several components of the position and the momentum operator, which are implied by the nontrivial commutation relations between the several components of the tensor  $\alpha^\mu_\nu$ , which are given by

$$[\alpha^{\mu\nu}, \alpha^{\rho\sigma}] = \Lambda^{\mu\nu\rho\sigma}, \quad (11)$$

where  $\Lambda^{\mu\nu\rho\sigma}$  is a tensor of fourth order, which can be represented as a matrix,

$$\Lambda^{\mu\nu\rho\sigma} = \varkappa \begin{pmatrix} \lambda_i^{\mu\nu} & \lambda_j^{\mu\nu} & \lambda_j^{\mu\nu} & \lambda_j^{\mu\nu} \\ \lambda_j^{\mu\nu} & \lambda_i^{\mu\nu} & \lambda_j^{\mu\nu} & \lambda_j^{\mu\nu} \\ \lambda_j^{\mu\nu} & \lambda_j^{\mu\nu} & \lambda_i^{\mu\nu} & \lambda_j^{\mu\nu} \\ \lambda_j^{\mu\nu} & \lambda_j^{\mu\nu} & \lambda_j^{\mu\nu} & \lambda_i^{\mu\nu} \end{pmatrix}, \quad (12)$$

containing the tensors  $\lambda_i^{\mu\nu}$  and  $\lambda_j^{\mu\nu}$  of second order, which are of the following shape, if they are again represented as matrices:

$$\lambda_i^{\mu\nu} = \begin{pmatrix} 0 & 2k & 2k & 2k \\ 2k & 0 & 2k & 2k \\ 2k & 2k & 0 & 2k \\ 2k & 2k & 2k & 0 \end{pmatrix}, \quad \lambda_j^{\mu\nu} = \begin{pmatrix} -2k & 0 & 0 & 0 \\ 0 & -2k & 0 & 0 \\ 0 & 0 & -2k & 0 \\ 0 & 0 & 0 & -2k \end{pmatrix}, \quad (13)$$

where  $k$  denotes according to (2) besides  $i$  and  $j$  the third imaginary unit in the quaternionic number space. The generalized quantization condition (7) is fulfilled by the following shape of the position and the momentum operator represented in position space:

$$\hat{x}^\mu = x^\mu, \quad \hat{p}_\mu = -\alpha^\nu_\mu \frac{\partial}{\partial x^\nu}. \quad (14)$$

The position and the momentum operator represented in momentum space are of the following shape:

$$\hat{x}^\mu = \alpha^\mu_\nu \frac{\partial}{\partial p_\nu}, \quad \hat{p}^\mu = p^\mu. \quad (15)$$

By using the representation of the momentum operator in position space given in (14) and the representation of the position operator in momentum operator given in (15), the commutation relations of the several components of the momentum operator with each other and of the several components of the position operator with each other can be determined. The position representation is only valid, if the algebra (7) is supplemented by the following commutation relations:

$$[\hat{x}^\mu, \hat{x}^\nu] = 0, \quad [\hat{p}^\mu, \hat{p}^\nu] = [\alpha^{\mu\rho}, \alpha^{\nu\sigma}] \frac{\partial}{\partial x^\rho} \frac{\partial}{\partial x^\sigma} = \Lambda^{\mu\rho\nu\sigma} \frac{\partial}{\partial x^\rho} \frac{\partial}{\partial x^\sigma} = \Lambda^{\mu\rho\nu\sigma} (\alpha^{-1})_{\rho\lambda} (\alpha^{-1})_{\sigma\kappa} \hat{p}^\lambda \hat{p}^\kappa, \quad (16)$$

and analogously, the momentum representation is only valid, if the algebra (7) is supplemented by the following commutation relations:

$$[\hat{x}^\mu, \hat{x}^\nu] = [\alpha^{\mu\rho}, \alpha^{\nu\sigma}] \frac{\partial}{\partial p^\rho} \frac{\partial}{\partial p^\sigma} = \Lambda^{\mu\rho\nu\sigma} \frac{\partial}{\partial p^\rho} \frac{\partial}{\partial p^\sigma} = \Lambda^{\mu\rho\nu\sigma} (\alpha^{-1})_{\rho\lambda} (\alpha^{-1})_{\sigma\kappa} \hat{x}^\lambda \hat{x}^\kappa, \quad [\hat{p}^\mu, \hat{p}^\nu] = 0. \quad (17)$$

In (16) and (17), the commutation relations between the several components of the quantization tensor  $\alpha^\mu_\nu$ , (11), has been used. The squared four-momentum operator as well as the squared four-position operator are equal to the corresponding squared operators of usual quantum mechanics and thus the following commutation relations are valid:

$$[\hat{p}^\mu \hat{p}_\mu, \hat{p}_\nu] = 0, \quad [\hat{x}^\mu \hat{x}_\mu, \hat{x}_\nu] = 0. \quad (18)$$

Of course, the commutation relations between the components of the corresponding angular momentum take a generalized form as well. The angular momentum operator is defined as

$$\hat{L}^a = \frac{1}{2} \epsilon^{abc} (\hat{x}_b \hat{p}_c + \hat{p}_c \hat{x}_b), \quad (19)$$

where  $\epsilon^{abc}$  denotes the total antisymmetric tensor in three dimensions. By using (7) as well as (16) or (17) respectively, the commutators between the components of the angular momentum operator with each other in case of the position representation as well as the momentum representation can be calculated and are given by

$$\begin{aligned} [\hat{L}^a, \hat{L}^d] = & \frac{1}{4} \epsilon^{abc} \epsilon^{def} \left[ 2\alpha_{bf} (\hat{x}_e \hat{p}_c + \hat{p}_c \hat{x}_e) - 2\alpha_{ec} (\hat{x}_b \hat{p}_f + \hat{p}_f \hat{x}_b) + \Lambda_{cgfh} (\alpha^{-1})^{gi} (\alpha^{-1})^{hj} \hat{x}_b \hat{x}_e \hat{p}_i \hat{p}_j \right. \\ & \left. + \Lambda_{cgfh} (\alpha^{-1})^{gi} (\alpha^{-1})^{hj} \hat{x}_b \hat{p}_i \hat{p}_j \hat{x}_e + \Lambda_{cgfh} (\alpha^{-1})^{gi} (\alpha^{-1})^{hj} \hat{x}_e \hat{p}_i \hat{p}_j \hat{x}_b + \Lambda_{cgfh} (\alpha^{-1})^{gi} (\alpha^{-1})^{hj} \hat{p}_i \hat{p}_j \hat{x}_b \hat{x}_e \right] \end{aligned} \quad (20)$$

in case of (16) corresponding to the position representation, and by

$$\begin{aligned} [\hat{L}^a, \hat{L}^d] = & \frac{1}{4} \epsilon^{abc} \epsilon^{def} \left[ 2\alpha_{bf} (\hat{x}_e \hat{p}_c + \hat{p}_c \hat{x}_e) - 2\alpha_{ec} (\hat{x}_b \hat{p}_f + \hat{p}_f \hat{x}_b) + \Lambda_{bgeh} (\alpha^{-1})^{gi} (\alpha^{-1})^{hj} \hat{x}_i \hat{x}_j \hat{p}_e \hat{p}_f \right. \\ & \left. + \Lambda_{bgeh} (\alpha^{-1})^{gi} (\alpha^{-1})^{hj} \hat{p}_f \hat{x}_i \hat{x}_j \hat{p}_e + \Lambda_{bgeh} (\alpha^{-1})^{gi} (\alpha^{-1})^{hj} \hat{p}_e \hat{x}_i \hat{x}_j \hat{p}_f + \Lambda_{bgeh} (\alpha^{-1})^{gi} (\alpha^{-1})^{hj} \hat{p}_e \hat{p}_f \hat{x}_i \hat{x}_j \right] \end{aligned} \quad (21)$$

in case of (17) corresponding to the momentum representation. The states  $|\psi\rangle$  of the Hilbert space  $\mathcal{H}_Q$ , where the operators of the quaternionic generalized quantum mechanics according to (7) live in, can be represented as wave-functions, which take quaternionic values,

$$\psi(x) = \psi_1(x) \mathbf{1} + \psi_i(x) i + \psi_j(x) j + \psi_k(x) k. \quad (22)$$

The inner product between two states constituting the Hilbert space  $\mathcal{H}_Q$ ,  $\langle \cdot | \cdot \rangle$ , represented in position space, can be defined in complete analogy to usual quantum mechanics to be

$$\langle \varphi | \psi \rangle = \int d^3x \varphi^*(x) \psi(x), \quad (23)$$

where  $\psi^*(x)$  denotes the conjugated quaternionic wave function, which is defined by (22) and the definition of quaternionic conjugation (3). Also hermitian conjugation of operators is defined in analogy to usual quantum mechanics by replacing complex conjugation by quaternionic conjugation. Accordingly the hermitian conjugated operator to an operator  $\hat{A}$  is defined as

$$\hat{A}^\dagger = \hat{A}^{*T}, \quad (24)$$

where the  $*$  again denotes quaternionic conjugation as specified in (3) and generalized hermiticity and unitarity are accordingly defined by the conditions  $A^\dagger = A$  and  $U^\dagger = U^{-1}$ . The generalized position and the generalized momentum operators are still hermitian operators. To show this, as usual one uses the relation  $\langle \varphi | \hat{A} | \psi \rangle = \langle \psi | \hat{A}^\dagger | \varphi \rangle^*$  with respect to the quaternionic case referring to (23) and (24), which implies for hermitian operators:  $\langle \varphi | \hat{A} | \psi \rangle = \langle \psi | \hat{A} | \varphi \rangle^*$ . Concerning the position operator this means

$$\begin{aligned} \langle \varphi | \hat{p}_\mu | \psi \rangle &= \int d^3x \varphi^*(x) \left( -\alpha^\nu_\mu \frac{\partial}{\partial x^\nu} \right) \psi(x) = \int d^3x \psi(x) \left( \alpha^\nu_\mu \frac{\partial}{\partial x^\nu} \right) \varphi^*(x) \\ &= \int d^3x \left[ \psi^*(x) \left( -\alpha^\nu_\mu \frac{\partial}{\partial x^\nu} \right) \varphi(x) \right]^* = \langle \psi | \hat{p}_\mu | \varphi \rangle^*. \end{aligned} \quad (25)$$

In (25) has been performed partial integration in the second step, where has been used that any wave-function  $\psi(x)$  representing a physical state  $|\psi\rangle$  has to go to zero at infinity to maintain that the function is square integrable  $\int d^3x |\psi|^2 < \infty$ , and that  $(\alpha^*)^\nu_\mu = -\alpha^\nu_\mu$ . The hermiticity of the position operator can of course be shown analogously. The generalized Klein-Gordon equation corresponding to the generalized representation of the momentum operators (14) looks as follows:

$$(\hat{p}^\mu \hat{p}_\mu - m^2) \psi(x) = 0 \quad \Leftrightarrow \quad (\alpha^{\mu\nu} \alpha_{\mu\rho} \partial_\nu \partial^\rho - m^2) \psi(x) = 0, \quad (26)$$

and the corresponding generalized Dirac equation looks as follows:

$$(\gamma^\mu \hat{p}_\mu + m) \psi = 0 \quad \Leftrightarrow \quad (\alpha^{\mu\nu} \gamma_\mu \partial_\nu + m) \psi = 0. \quad (27)$$

The corresponding Dirac Lagrangian to (27) is of the following form:

$$\mathcal{L} = \bar{\psi} (\alpha^{\mu\nu} \gamma_\mu \partial_\nu - m) \psi, \quad (28)$$

where  $\bar{\psi} = \psi^\dagger \gamma^0$  and the  $\dagger$  denotes quaternionic adjungation according to (24). It is important to mention that the Dirac matrices denoted by  $\gamma^\mu$  referring to the Dirac spinor space are still formulated with usual complex numbers and accordingly they commute with the quaternionic quantization tensor,

$$[\alpha^{\mu\nu}, \gamma^\rho] = 0. \quad (29)$$

The solution of the generalized Dirac equation (27) as well as of the corresponding generalized Klein-Gordon equation (26) is defined by the generalized plane waves, which are equivalent to the eigenstates of the momentum operator in position space,  $|p\rangle$ , which are given by

$$|p\rangle = \exp \left[ - (\alpha^{-1})^{\mu\nu} p_\mu x_\nu \right], \quad (30)$$

what can be seen by applying the momentum operator to  $|p\rangle$ :

$$\begin{aligned} \hat{p}_\mu |p\rangle &= -\alpha^\nu_\mu \partial_\nu \exp \left[ - (\alpha^{-1})^{\rho\sigma} p_\rho x_\sigma \right] = -\alpha^\nu_\mu \partial_\nu \left[ (-\alpha^{-1})^{\rho\sigma} p_\rho x_\sigma \right] \exp \left[ - (\alpha^{-1})^{\rho\sigma} p_\rho x_\sigma \right] \\ &= \alpha^\nu_\mu (\alpha^{-1})^\rho_\nu p_\rho \exp \left[ - (\alpha^{-1})^{\rho\sigma} p_\rho x_\sigma \right] = \delta^\rho_\mu p_\rho \exp \left[ - (\alpha^{-1})^{\rho\sigma} p_\rho x_\sigma \right] \\ &= p_\mu \exp \left[ - (\alpha^{-1})^{\rho\sigma} p_\rho x_\sigma \right] = p_\mu |p\rangle. \end{aligned} \quad (31)$$

To separate the components belonging to  $i$  and to  $j$ , the momentum eigenstates (30) can be rewritten to

$$\begin{aligned} \exp \left[ - (\alpha^{-1})^{\mu\nu} p_\mu x_\nu \right] &= \exp \left[ - (a_i i p^\mu x_\mu + a_j j p^\mu x_\mu + b_i i p_0 x_1 + b_j j p_0 x_1 + b_i i p_0 x_2 + b_j j p_0 x_2 + b_i i p_0 x_3 + b_j j p_0 x_3 \right. \\ &\quad + b_i i p_1 x_0 + b_j j p_1 x_0 + b_i i p_1 x_2 + b_j j p_1 x_2 + b_i i p_1 x_3 + b_j j p_1 x_3 \\ &\quad + b_i i p_2 x_0 + b_j j p_2 x_0 + b_i i p_2 x_1 + b_j j p_2 x_1 + b_i i p_2 x_3 + b_j j p_2 x_3 \\ &\quad \left. + b_i i p_3 x_0 + b_j j p_3 x_0 + b_i i p_3 x_1 + b_j j p_3 x_1 + b_i i p_3 x_2 + b_j j p_3 x_2) \right] \\ &= \exp \left[ - (i \beta_i^{\mu\nu} p_\mu x_\nu + j \beta_j^{\mu\nu} p_\mu x_\nu) \right], \end{aligned} \quad (32)$$

where the tensors  $\beta_i^{\mu\nu}$  as well as  $\beta_j^{\mu\nu}$  have been defined, which can be represented as matrices in the following way:

$$\beta_i^{\mu\nu} = \begin{pmatrix} a_i & b_i & b_i & b_i \\ b_i & a_i & b_i & b_i \\ b_i & b_i & a_i & b_i \\ b_i & b_i & b_i & a_i \end{pmatrix}, \quad \beta_j^{\mu\nu} = \begin{pmatrix} a_j & b_j & b_j & b_j \\ b_j & a_j & b_j & b_j \\ b_j & b_j & a_j & b_j \\ b_j & b_j & b_j & a_j \end{pmatrix}. \quad (33)$$

Remember that the entries  $a_i, a_j, b_i$  and  $b_j$  of the matrix representation of the tensors  $\beta_i^{\mu\nu}$  and  $\beta_j^{\mu\nu}$  in (33), have already been defined in (10). Since a complete set of eigenstates to all four components of the momentum operator in position space can be found (30),(31), although the components of the momentum operator do not commute with each other in case of the position representation (16), the theorem that such a set of eigenstates with respect to two operators does exist exactly then, if these commutators commute with each other, does not hold anymore in the presented quaternionic generalization of quantum mechanics. This property of course arises directly from the noncommutativity of the units of the imaginary directions,  $i, j, k$ , of the quaternionic number space.

#### IV. CONSEQUENCES FOR QUANTUM FIELD THEORY

##### A. Calculation of the Generalized Propagator

In this section are considered the corresponding consequences of the generalized quantization postulate of quantum mechanics (7), which has been introduced in the last section, for quantum field theory. Since with respect to the derivation of the propagator a scalar field is considered, the generalized quantization postulate has not to be transferred to field quantization here, but the influence arising from the generalized quantum theoretical field equations to quantum field theory are explored. By using the generalized expression of the plane waves as momentum eigenstates given in (30) and (32) respectively, a free scalar field as solution of the generalized Klein-Gordon equation reads

$$\phi(x) = \int \frac{d^3 p}{\sqrt{(2\pi)^3 2p_0}} \{ q \exp [-i \beta_i^{\mu\nu} p_\mu x_\nu - j \beta_j^{\mu\nu} p_\mu x_\nu] + q^* \exp [i \beta_i^{\mu\nu} p_\mu x_\nu + j \beta_j^{\mu\nu} p_\mu x_\nu] \}. \quad (34)$$

The quaternionic quantization principle of quantum mechanics (7) influences the shape of free quantum theoretical field equations and thus the plane waves, but it does not influence the Fock space structure of the Hilbert space of many particles. Therefore the postulated commutation relations between the coefficients of the plane waves concerning field quantization remain the same. This means that to obtain the quantum properties of a scalar field in case of quaternionic quantization, the coefficients have to become operators,  $q \rightarrow \hat{q}, q^* \rightarrow \hat{q}^\dagger$ , which have the same properties as in the usual case,

$$[\hat{q}(p), \hat{q}^\dagger(p')] = \delta(p - p'). \quad (35)$$

The corresponding scalar field operator reads

$$\hat{\phi}(x) = \int \frac{d^3 p}{\sqrt{(2\pi)^3 2p_0}} \{ \hat{q} \exp [-i \beta_i^{\mu\nu} p_\mu x_\nu - j \beta_j^{\mu\nu} p_\mu x_\nu] + \hat{q}^\dagger \exp [i \beta_i^{\mu\nu} p_\mu x_\nu + j \beta_j^{\mu\nu} p_\mu x_\nu] \}. \quad (36)$$

To obtain the corresponding propagator to this generalized quantum field (36), as usual one has to consider the expectation value with respect to the vacuum state,  $|0\rangle$ , of the time ordered product,

$$T[A(t_1)B(t_2)] = \begin{cases} A(t_1)B(t_2), & \text{if } t_1 > t_2 \\ B(t_2)A(t_1), & \text{if } t_2 > t_1 \end{cases}, \quad (37)$$

of the field operator at two different space-time points  $x$  and  $y$ ,

$$\begin{aligned} G(x-y) &= \langle 0|T[\hat{\phi}(x)\hat{\phi}(y)]|0\rangle \\ &= \int \frac{d^3p}{(2\pi)^3 2p_0} \{ \theta(x_0 - y_0) \exp[\mathcal{Z}(-\kappa(p, x), \kappa(p, y))] + \theta(y_0 - x_0) \exp[\mathcal{Z}(-\kappa(p, y), \kappa(p, x))] \} \\ &= i \int \frac{d\tau}{2\pi} \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{\exp[-i\tau(x_0 - y_0)] \exp[\mathcal{Z}(-\kappa(p, x), \kappa(p, y))]}{2p_0\tau + i\epsilon} + \frac{\exp[-i\tau(y_0 - x_0)] \exp[\mathcal{Z}(-\kappa(p, y), \kappa(p, x))]}{2p_0\tau + i\epsilon} \right\} \\ &= i \int \frac{d\tau}{2\pi} \int \frac{d^3p}{(2\pi)^3} \frac{\exp[-i\tau(x_0 - y_0)]}{2p_0\tau + i\epsilon} \{ \exp[\mathcal{Z}(-\kappa(p, x), \kappa(p, y))] - \exp[\mathcal{Z}(-\kappa(p, y), \kappa(p, x))] \}, \end{aligned} \quad (38)$$

where has been defined

$$\kappa(p, x) = i\beta_i^{\mu\nu} p_\mu x_\nu + j\beta_j^{\mu\nu} p_\mu x_\nu. \quad (39)$$

In (38) have been used as usual the definition of the  $\Theta$ -function,

$$\Theta(x_0 - y_0) = \lim_{\epsilon \rightarrow 0} -\frac{1}{2\pi i} \int d\tau \frac{\exp[-i\tau(x_0 - y_0)]}{\tau + i\epsilon}, \quad (40)$$

and the Baker-Campbell-Hausdorff formula,

$$\exp A \exp B = \exp \mathcal{Z}(A, B), \quad \text{with} \quad \mathcal{Z}(A, B) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{i=1}^n \sum_{r_i + s_i > 0} \prod_{m=1}^n \frac{A^{r_m} B^{s_m}}{r_m! s_m!}. \quad (41)$$

## B. Quaternionic Gauge Principle

The wave function in the generalized Dirac equation (27) represents a quaternionic spinor wave function. This means that the spinor structure is the same as in usual quantum field theory, but the wave function is quaternionic and thus it is of a shape as defined in (22). With respect to usual complex wave functions one can perform phase transformation,  $\psi_{\mathbb{C}}(x) \rightarrow e^{i\alpha} \psi_{\mathbb{C}}(x)$ . The free wave equations are invariant under such a transformation, if it is chosen to be a global transformation. The postulate of local invariance under a phase transformation leads to the necessity to introduce the electromagnetic potential and thus the electromagnetic interaction has its origin in a symmetry principle. This symmetry principle has to be generalized in the quaternionic case, since in this case a transformation is possible, which refers to all quaternionic directions,  $i$ ,  $j$  and  $k$ . The intricacy of such a generalization consists in the noncommutativity between the quantities  $i$ ,  $j$  and  $k$ , which build a Lie Algebra, the Lie Algebra belonging to the  $SU(2)$  namely. Accordingly even usual electrodynamics has to be generalized to a certain kind of non Abelian gauge theory. The quaternionic Dirac equation (27) contains the following generalized quaternionic phase invariance:

$$\psi \longrightarrow \exp(i\varphi + \varkappa j\chi + \varkappa k\rho)\psi, \quad \alpha^{\mu\nu} \longrightarrow \exp(i\varphi + \varkappa j\chi + \varkappa k\rho)\alpha^{\mu\nu} \exp(-i\varphi - \varkappa j\chi - \varkappa k\rho). \quad (42)$$

The prefactor  $\varkappa$  of the quantities  $j$  and  $k$  maintains that the theory becomes approximatively equal to usual electrodynamics, if  $\varkappa$  goes to zero. Since  $\alpha^{\mu\nu}$  is a quaternionic tensor, it has of course to be transformed as well. A precondition for the invariance of the quaternionic Dirac equation (27) under the generalized quaternionic phase transformations (42) is that the  $\gamma$ -matrices commute with the quaternionic quantization tensor  $\alpha^{\mu\nu}$  (29). If this symmetry



is postulated to be a local symmetry, then the corresponding potential as generalization of the usual electromagnetic potential has to contain three components referring to the three directions of the quaternionic space. This means that the Lagrangian containing the local quaternionic phase invariance reads as follows:

$$\mathcal{L} = \bar{\psi} (\alpha^{\mu\nu} \gamma_\mu \mathcal{D}_\nu - m) \psi, \quad (43)$$

where the covariant derivative  $\mathcal{D}_\mu$  is defined in the following way:

$$\mathcal{D}_\mu = \partial_\mu + i\mathcal{A}_\mu + \varkappa j\mathcal{B}_\mu + \varkappa k\mathcal{C}_\mu. \quad (44)$$

The gauge potentials appearing in (44) under an infinitesimal transformation have to transform according to

$$\begin{aligned} \mathcal{A}_\mu &\longrightarrow \mathcal{A}_\mu - \partial_\mu \varphi - \varkappa k\mathcal{A}_\mu \chi + \varkappa j\mathcal{A}_\mu \rho, \\ \mathcal{B}_\mu &\longrightarrow \mathcal{B}_\mu - \partial_\mu \chi - \varkappa i\mathcal{B}_\mu \rho + k\mathcal{B}_\mu \varphi, \\ \mathcal{C}_\mu &\longrightarrow \mathcal{C}_\mu - \partial_\mu \rho - j\mathcal{C}_\mu \varphi + \varkappa i\mathcal{C}_\mu \chi, \end{aligned} \quad (45)$$

if a local phase transformation is considered. This means that the Lagrangian (43) is invariant under combined local transformations of the shape (42) and (45) and represents the generalization of electromagnetism with respect to the quaternionic generalization of quantum mechanics according to this paper. Accordingly also a generalized field strength tensor has to be built based on the generalized covariant derivative (44), which is as usual defined as commutator of the components of the covariant derivative and accordingly reads

$$\begin{aligned} \mathcal{F}_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu] &= i [\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + \varkappa^2 (\mathcal{B}_\mu \mathcal{C}_\nu - \mathcal{C}_\mu \mathcal{B}_\nu)] + \varkappa j [\partial_\mu \mathcal{B}_\nu - \partial_\nu \mathcal{B}_\mu + \mathcal{C}_\mu \mathcal{A}_\nu - \mathcal{A}_\mu \mathcal{C}_\nu] \\ &\quad + \varkappa k [\partial_\mu \mathcal{C}_\nu - \partial_\nu \mathcal{C}_\mu + \mathcal{A}_\mu \mathcal{B}_\nu - \mathcal{B}_\mu \mathcal{A}_\nu] \\ &\equiv i\mathcal{I}_{\mu\nu} + \varkappa j\mathcal{J}_{\mu\nu} + \varkappa k\mathcal{K}_{\mu\nu}, \end{aligned} \quad (46)$$

where the last line (46) serves as a definition of the several components  $\mathcal{I}_{\mu\nu}$ ,  $\mathcal{J}_{\mu\nu}$  and  $\mathcal{K}_{\mu\nu}$ . In analogy to the usual case it is suggesting to postulate the following Lagrangian for the interaction fields of the generalized electrodynamics,

$$\mathcal{L} = \frac{1}{4} \mathcal{F}_{\mu\nu}^* \mathcal{F}^{\mu\nu} = \frac{1}{4} \mathcal{I}_{\mu\nu} \mathcal{I}^{\mu\nu} + \frac{1}{4} \varkappa^2 \mathcal{J}_{\mu\nu} \mathcal{J}^{\mu\nu} + \frac{1}{4} \varkappa^2 \mathcal{K}_{\mu\nu} \mathcal{K}^{\mu\nu}. \quad (47)$$

## V. CANONICAL QUANTIZATION OF GENERAL RELATIVITY WITH QUATERNIONS

### A. Quaternionic Quantization in Quantum Geometrodynamics

In the last two sections the quaternionic quantization principle has been considered with respect to quantum mechanics and the corresponding consequences for quantum field theory. But the main interest of this extension of quantum theory arises, if it is explored within the quantum description of general relativity. Since the appropriate quantum description of general relativity has not been found yet, it could indeed be possible that the generalization of the concept of quantization, which is suggested in this paper, is necessary to incorporate also general relativity to a quantum description of all interactions, although the approximation with  $\varkappa \rightarrow 0$  was appropriate to treat the physics of elementary particles at low energies, which is based on the other fundamental interactions in nature. Of course, the generalization of the quantization concept has been formulated with respect to canonical quantization. Accordingly, a modification of the canonical quantization of general relativity is considered in this section and in the next section the corresponding modification is extended to canonical quantum supergravity. The canonical quantization of general relativity is based on a foliation of space-time into a spacelike three dimensional submanifold  $\Sigma$  and one separated space-time direction described by  $\tau$ , which is considered as time-coordinate. Accordingly the metric  $g_{\mu\nu}$  referring to the complete space-time can be splitted into a part referring to the spacelike submanifold  $\Sigma$ , which is denoted as  $h_{ab}$ , and the other components, which are related to the time coordinate  $\tau$  and are expressed by the lapse function  $N$  and the shift vector  $N_a$ , see [97] for example. The complete metric expressed by these variables reads as follows:

$$g_{\mu\nu} = \begin{pmatrix} N_a N^a - N^2 & N_b \\ N_c & h_{ab} \end{pmatrix}. \quad (48)$$

If the representation (48) of the metric is used, then the Einstein-Hilbert action can be written in the following way:

$$S_{EH} = \int_{\mathcal{M}} dt d^3x \mathcal{L}_g = \frac{1}{16\pi G} \int_{\mathcal{M}} dt d^3x N \left( G^{abcd} K_{ab} K_{cd} + \sqrt{h} [R_h - 2\Lambda] \right), \quad (49)$$

where  $G$  denotes the gravitational constant,  $R_h$  denotes the part of the Ricci scalar built from the three metric  $h_{ab}$  and thus refers to the submanifold  $\Sigma$  and  $K_{ab}$  denotes the extrinsic curvature, which is defined as

$$K_{ab} = \frac{1}{2N} \left( \dot{h}_{ab} - D_a N_b - D_b N_a \right), \quad (50)$$

and the forth order tensor  $G_{abcd}$  is called DeWitt metric and is defined as

$$G_{abcd} = \frac{1}{2\sqrt{h}} (h_{ac} h_{bd} + h_{ad} h_{bc} - h_{ab} h_{cd}). \quad (51)$$

By referring to the Lagrangian  $\mathcal{L}_g$  within (49), the canonical conjugated quantity  $\pi^{ab}$  can be defined,

$$\pi^{ab} = \frac{\partial \mathcal{L}_g}{\partial \dot{h}_{ab}} = \frac{\sqrt{h}}{16\pi G} (K^{ab} - K h^{ab}), \quad (52)$$

and by using the canonical conjugated momentum (52) the Einstein-Hilbert action (49) can be reexpressed to

$$S_{EH} = \frac{1}{16\pi G} \int_{\mathcal{M}} dt d^3x \left( \pi^{ab} \dot{h}_{ab} - N \mathcal{H}_\tau - N^a \mathcal{H}_a \right), \quad (53)$$

where  $\mathcal{H}_\tau$  denotes the part of the Hamiltonian density referring to the time direction  $\tau$  and  $\mathcal{H}_a$  denotes the part of the Hamiltonian density referring to the submanifold  $\Sigma$ . Variation of (53) with respect to the three metric yields the dynamical constraints, the Hamiltonian constraint and the diffeomorphism constraint,

$$\mathcal{H}_\tau = 16\pi G G_{abcd} \pi^{ab} \pi^{cd} - \frac{\sqrt{h}}{16\pi G} (R_h - 2\Lambda) = 0, \quad \mathcal{H}_a = -2D_b \pi^b_a = 0. \quad (54)$$

To obtain a quantum description of canonical general relativity, the three metric  $h_{ab}$  defined in (48) as well as the corresponding canonical conjugated variable  $\pi_{ab}$  defined in (52) have to be converted to operators,  $h_{ab} \rightarrow \hat{h}_{ab}$  and  $\pi^{ab} \rightarrow \hat{\pi}^{ab}$ . Within the usual description of quantum geometrodynamics one postulates in analogy to the usual Heisenbergian commutation relation between position and momentum as quantization principle in quantum mechanics the following commutation relations between the operator describing the three metric  $\hat{h}_{ab}$  and the operator describing the corresponding canonical conjugated momentum  $\hat{\pi}^{ab}$ ,

$$\left[ \hat{h}_{ab}(x), \hat{\pi}^{cd}(y) \right] = \frac{i}{2} \left( \delta_a^c \delta_b^d + \delta_b^c \delta_a^d \right) \delta(x - y). \quad (55)$$

If now the assumption is made that general quantum theory contains a quaternionic quantization principle, then this quantization principle becomes not only manifest with respect to quantum mechanics, but also a quantum description of general relativity has to be based on a corresponding quantization. This means that the quaternionic quantization principle (7) as it has been formulated as generalization of quantum mechanics has to be transferred to the variables of canonical quantum gravity and thus (55) has to be generalized. To be analogue to the case of quantum mechanics, the generalization has to be performed in such a way that the commutation relations between the components of the three metric  $\hat{h}_{ab}$  and the corresponding components of the canonical conjugated variable  $\hat{\pi}^{ab}$  remain the same and the commutation relations between the components of  $\hat{h}_{ab}$  and all the other components of  $\hat{\pi}^{ab}$  variable are equal to  $j$  times the parameter  $\varkappa$ . This leads to the following transition of the commutation relation between  $\hat{h}_{ab}$  and  $\hat{\pi}^{ab}$ :

$$\left[ \hat{h}_{ab}(x), \hat{\pi}^{cd}(y) \right] = \frac{i}{2} \left( \delta_a^c \delta_b^d + \delta_b^c \delta_a^d \right) \delta(x - y) \quad \longrightarrow \quad \left[ \hat{h}_{ab}(x), \hat{\pi}^{cd}(y) \right] = M_{ab}^{cd} \delta(x - y), \quad (56)$$

where has been introduced the new quaternionic tensor  $M_{ab}^{cd}$  of fourth order in three dimensions, which can be written as a three cross three matrix, which contains second order tensors as entries of this matrix,

$$M_{ab}^{cd} = \frac{1}{2} \begin{pmatrix} m_{a1}^{c1} & m_{a1}^{c2} & m_{a1}^{c3} \\ m_{a2}^{c1} & m_{a2}^{c2} & m_{a2}^{c3} \\ m_{a3}^{c1} & m_{a3}^{c2} & m_{a3}^{c3} \end{pmatrix}. \quad (57)$$

The entries of (57) can again be written as three cross three matrices, which contain quaternionic expressions and are of the following form:

$$\begin{aligned} m_{a1}^{c1} &= \begin{pmatrix} 2i & 2\kappa j & 2\kappa j \\ 2\kappa j & i & \kappa j \\ 2\kappa j & \kappa j & i \end{pmatrix}, & m_{a1}^{c2} &= \begin{pmatrix} 2\kappa j & 2\kappa j & 2\kappa j \\ i & 2\kappa j & \kappa j \\ \kappa j & 2\kappa j & \kappa j \end{pmatrix}, & m_{a1}^{c3} &= \begin{pmatrix} 2\kappa j & 2\kappa j & 2\kappa j \\ \kappa j & \kappa j & 2\kappa j \\ i & \kappa j & 2\kappa j \end{pmatrix}, \\ m_{a2}^{c1} &= \begin{pmatrix} 2\kappa j & i & \kappa j \\ 2\kappa j & 2\kappa j & 2\kappa j \\ 2\kappa j & \kappa j & \kappa j \end{pmatrix}, & m_{a2}^{c2} &= \begin{pmatrix} i & 2\kappa j & \kappa j \\ 2\kappa j & 2i & 2\kappa j \\ \kappa j & 2\kappa j & i \end{pmatrix}, & m_{a2}^{c3} &= \begin{pmatrix} \kappa j & \kappa j & 2\kappa j \\ 2\kappa j & 2\kappa j & 2\kappa j \\ \kappa j & i & 2\kappa j \end{pmatrix}, \\ m_{a3}^{c1} &= \begin{pmatrix} 2\kappa j & \kappa j & i \\ 2\kappa j & \kappa j & \kappa j \\ 2\kappa j & 2\kappa j & 2\kappa j \end{pmatrix}, & m_{a3}^{c2} &= \begin{pmatrix} \kappa j & 2\kappa j & \kappa j \\ \kappa j & 2\kappa j & i \\ 2\kappa j & 2\kappa j & 2\kappa j \end{pmatrix}, & m_{a3}^{c3} &= \begin{pmatrix} i & \kappa j & 2\kappa j \\ \kappa j & i & 2\kappa j \\ 2\kappa j & 2\kappa j & 2i \end{pmatrix}. \end{aligned} \quad (58)$$

As in the usual case the factor  $\frac{1}{2}$  of some entries arises from the symmetry property of the three metric,  $h_{ab} = h_{ba}$  leading also to  $\pi^{ab} = \pi^{ba}$ , implying that the components of the components with different indices would appear doubly, if the corresponding factor two would not be removed. As usual the operators act on states  $|\Psi\rangle$ , which are functionals depending on  $h_{ab}$  or  $\pi^{ab}$  respectively, but take in analogy to the generalized states in quantum mechanics quaternionic values. If the three metric representation of the operators defined by the quaternionic quantization principle of general relativity is used (56), these operators read

$$\hat{h}_{ab}(x) |\Psi[h(x)]\rangle = h_{ab}(x) |\Psi[h(x)]\rangle, \quad \hat{\pi}^{ab}(x) |\Psi[h(x)]\rangle = -M_{cd}^{ab} \frac{\delta}{\delta h_{cd}(x)} |\Psi[h(x)]\rangle. \quad (59)$$

The quaternionic quantization principle implies also nontrivial commutation relations between the components of the three metric operator  $\hat{h}_{ab}$  and the components of the operator of the canonical conjugated quantity  $\hat{\pi}^{ab}$ . The commutation relations between the components of the three metric operator  $\hat{h}_{ab}$  can be calculated by using the representation with respect to the canonical conjugated quantity reading as follows:

$$\hat{h}_{ab}(x) |\Psi[\pi(x)]\rangle = M_{ab}^{cd} \frac{\delta}{\delta \pi^{cd}(x)} |\Psi[\pi(x)]\rangle, \quad \hat{\pi}^{ab}(x) |\Psi[\pi(x)]\rangle = \pi^{ab}(x) |\Psi[\pi(x)]\rangle. \quad (60)$$

The commutation relations between the components of the three metric arise from the fact that the components of the quaternionic tensor of fourth order,  $M_{cd}^{ab}$ , defining the quaternionic quantization principle (56) and being defined in (57) and (58) do not commute with each other as the components of the quaternionic tensor in quantum mechanics  $\alpha_{\nu}^{\mu}$  defined in (8). Therefore to perform the calculation, the commutation relations between the components of the quaternionic tensor of fourth order,  $M_{cd}^{ab}$ , are important. The components of  $M_{cd}^{ab}$  fulfil the following commutation relations with each other, which are analogue to the commutation relations referring to  $\alpha_{\nu}^{\mu}$  (11):

$$[M_{cd}^{ab}, M_{gh}^{ef}] = \mathcal{M}_{cdgh}^{abef}, \quad (61)$$

where  $\mathcal{M}_{cdgh}^{abef}$  is a tensor of eight order, which can be written as a matrix containing tensors of sixth order,

$$\mathcal{M}_{cdgh}^{abef} = \varkappa \begin{pmatrix} \mu_{c1gh}^{a1ef} & \mu_{c2gh}^{a1ef} & \mu_{c3gh}^{a1ef} \\ \mu_{c1gh}^{a2ef} & \mu_{c2gh}^{a2ef} & \mu_{c3gh}^{a2ef} \\ \mu_{c1gh}^{a3ef} & \mu_{c2gh}^{a3ef} & \mu_{c3gh}^{a3ef} \end{pmatrix}, \quad (62)$$

where the tensors appearing in the matrix can be written as matrices again,

$$\begin{aligned}
\mu_{a1ef}^{c1gh} = [m_{a1}^{c1}, m_{ef}^{gh}] &= \begin{pmatrix} 2m_{i\ ef}^{gh} & 2m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} \\ 2m_{j\ ef}^{gh} & m_{i\ ef}^{gh} & m_{j\ ef}^{gh} \\ 2m_{j\ ef}^{gh} & m_{j\ ef}^{gh} & m_{i\ ef}^{gh} \end{pmatrix}, & \mu_{a1ef}^{c2gh} = [m_{a1}^{c2}, m_{ef}^{gh}] &= \begin{pmatrix} 2m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} \\ m_{i\ ef}^{gh} & 2m_{j\ ef}^{gh} & m_{j\ ef}^{gh} \\ m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} & m_{j\ ef}^{gh} \end{pmatrix}, \\
\mu_{a1ef}^{c3gh} = [m_{a1}^{c3}, m_{ef}^{gh}] &= \begin{pmatrix} 2m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} \\ m_{j\ ef}^{gh} & m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} \\ m_{i\ ef}^{gh} & m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} \end{pmatrix}, & \mu_{a2ef}^{c1gh} = [m_{a2}^{c1}, m_{ef}^{gh}] &= \begin{pmatrix} 2m_{j\ ef}^{gh} & m_{i\ ef}^{gh} & m_{j\ ef}^{gh} \\ 2m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} \\ 2m_{j\ ef}^{gh} & m_{j\ ef}^{gh} & m_{j\ ef}^{gh} \end{pmatrix}, \\
\mu_{a2ef}^{c2gh} = [m_{a2}^{c2}, m_{ef}^{gh}] &= \begin{pmatrix} m_{i\ ef}^{gh} & 2m_{j\ ef}^{gh} & m_{j\ ef}^{gh} \\ 2m_{j\ ef}^{gh} & 2m_{i\ ef}^{gh} & 2m_{j\ ef}^{gh} \\ m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} & m_{i\ ef}^{gh} \end{pmatrix}, & \mu_{a2ef}^{c3gh} = [m_{a2}^{c3}, m_{ef}^{gh}] &= \begin{pmatrix} m_{j\ ef}^{gh} & m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} \\ 2m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} \\ m_{j\ ef}^{gh} & m_{i\ ef}^{gh} & 2m_{j\ ef}^{gh} \end{pmatrix}, \\
\mu_{a3ef}^{c1gh} = [m_{a3}^{c1}, m_{ef}^{gh}] &= \begin{pmatrix} 2m_{i\ ef}^{gh} & m_{j\ ef}^{gh} & m_{i\ ef}^{gh} \\ 2m_{j\ ef}^{gh} & m_{j\ ef}^{gh} & m_{j\ ef}^{gh} \\ 2m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} \end{pmatrix}, & \mu_{a3ef}^{c2gh} = [m_{a3}^{c2}, m_{ef}^{gh}] &= \begin{pmatrix} m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} & m_{j\ ef}^{gh} \\ m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} & m_{i\ ef}^{gh} \\ 2m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} \end{pmatrix}, \\
\mu_{a3ef}^{c3gh} = [m_{a3}^{c3}, m_{ef}^{gh}] &= \begin{pmatrix} m_{i\ ef}^{gh} & m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} \\ m_{j\ ef}^{gh} & m_{i\ ef}^{gh} & 2m_{j\ ef}^{gh} \\ 2m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} & 2m_{j\ ef}^{gh} \end{pmatrix}, & & (63)
\end{aligned}$$

which again contain tensors of fourth order, which entries are defined as

$$\begin{aligned}
m_{i\ a1}^{c1} = [i, m_{a1}^{c1}] &= \begin{pmatrix} 0 & 4k & 4k \\ 4k & 0 & 2k \\ 4k & 2k & 0 \end{pmatrix}, & m_{i\ a1}^{c2} = [i, m_{a1}^{c2}] &= \begin{pmatrix} 4k & 4k & 4k \\ 0 & 4k & 2k \\ 2k & 4k & 2k \end{pmatrix}, & m_{i\ a1}^{c3} = [i, m_{a1}^{c3}] &= \begin{pmatrix} 4k & 4k & 4k \\ 2k & 2k & 4k \\ 0 & 2k & 4k \end{pmatrix}, \\
m_{i\ a2}^{c1} = [i, m_{a2}^{c1}] &= \begin{pmatrix} 4k & 0 & 2k \\ 4k & 4k & 4k \\ 4k & 2k & 2k \end{pmatrix}, & m_{i\ a2}^{c2} = [i, m_{a2}^{c2}] &= \begin{pmatrix} 0 & 4k & 2k \\ 4k & 0 & 4k \\ 2k & 4k & 0 \end{pmatrix}, & m_{i\ a2}^{c3} = [i, m_{a2}^{c3}] &= \begin{pmatrix} 2k & 2k & 4k \\ 4k & 4k & 4k \\ 2k & 0 & 4k \end{pmatrix}, \\
m_{i\ a3}^{c1} = [i, m_{a3}^{c1}] &= \begin{pmatrix} 4k & 2k & 0 \\ 4k & 2k & 2k \\ 4k & 4k & 4k \end{pmatrix}, & m_{i\ a3}^{c2} = [i, m_{a3}^{c2}] &= \begin{pmatrix} 2k & 4k & 2k \\ 2k & 4k & 0 \\ 4k & 4k & 4k \end{pmatrix}, & m_{i\ a3}^{c3} = [i, m_{a3}^{c3}] &= \begin{pmatrix} 0 & 2k & 4k \\ 2k & 0 & 4k \\ 4k & 4k & 0 \end{pmatrix}, \\
m_{j\ a1}^{c1} = [j, m_{a1}^{c1}] &= \begin{pmatrix} -4k & 0 & 0 \\ 0 & -2k & 0 \\ 0 & 0 & -2k \end{pmatrix}, & m_{j\ a1}^{c2} = [j, m_{a1}^{c2}] &= \begin{pmatrix} 0 & 0 & 0 \\ -2k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & m_{j\ a1}^{c3} = [j, m_{a1}^{c3}] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2k & 0 & 0 \end{pmatrix}, \\
m_{j\ a2}^{c1} = [j, m_{a2}^{c1}] &= \begin{pmatrix} 0 & -2k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & m_{j\ a2}^{c2} = [j, m_{a2}^{c2}] &= \begin{pmatrix} -2k & 0 & 0 \\ 0 & -4k & 0 \\ 0 & 0 & -2k \end{pmatrix}, & m_{j\ a2}^{c3} = [j, m_{a2}^{c3}] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2k & 0 \end{pmatrix}, \\
m_{j\ a3}^{c1} = [j, m_{a3}^{c1}] &= \begin{pmatrix} 0 & 0 & -2k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & m_{j\ a3}^{c2} = [j, m_{a3}^{c2}] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2k \\ 0 & 0 & 0 \end{pmatrix}, & m_{j\ a3}^{c3} = [j, m_{a3}^{c3}] &= \begin{pmatrix} -2k & 0 & 0 \\ 0 & -2k & 0 \\ 0 & 0 & -4k \end{pmatrix}. & (64)
\end{aligned}$$

Accordingly the components of the three metric operator  $\hat{h}_{ab}$  fulfil the following commutation relations:

$$[\hat{h}_{ab}(x), \hat{h}_{cd}(y)] = \left[ -M_{ab}^{ef} \frac{\delta}{\delta \pi_{ef}(x)}, -M_{cd}^{gh} \frac{\delta}{\delta \pi_{gh}(y)} \right] = \mathcal{M}_{abcd}^{efgh} (M^{-1})_{ef}^{ij} (M^{-1})_{gh}^{kl} \hat{h}_{ij}(x) \hat{h}_{kl}(y). \quad (65)$$

The commutation relations between the components of the operator of the canonical conjugated quantity  $\hat{\pi}^{ab}$ , which can be calculated by referring to the three metric representation, read as follows

$$[\hat{\pi}_{ab}(x), \hat{\pi}_{cd}(y)] = \left[ M_{ab}^{ef} \frac{\delta}{\delta h_{ef}(x)}, M_{cd}^{gh} \frac{\delta}{\delta h_{gh}(y)} \right] = \mathcal{M}_{abcd}^{efgh} (M^{-1})_{ef}^{ij} (M^{-1})_{gh}^{kl} \hat{\pi}_{ij}(x) \hat{\pi}_{kl}(y). \quad (66)$$

It is remarkable that according to (65) and (66) the components of the operators  $\hat{h}_{ab}$  and  $\hat{\pi}^{ab}$  respectively fulfil even nontrivial commutation relations with the corresponding components of the operators at other space points, what induces a kind of nonlocality. This nonlocality has its origin in the property that the noncommutativity in (65) and (66) arises from the noncommutativity of the quaternionic components and this structure does not depend on the space point. To obtain the quantum constraints restricting the states, which are physically possible, the canonical variables  $h_{ab}$  and  $\pi^{ab}$  appearing in (108) have to be replaced by the corresponding operators  $\hat{h}_{ab}$  and  $\hat{\pi}^{ab}$ . This yields the following quaternionic Wheeler-DeWitt equation as quantum theoretical analogon to the Hamiltonian constraint as well the quantum theoretical version of the diffeomorphism constraint, which read as follows:

$$\left[ 16\pi G G_{abcd} M_{ef}^{ab} M_{gh}^{cd} \frac{\delta}{\delta h_{ef}} \frac{\delta}{\delta h_{gh}} - \frac{\sqrt{h}}{16\pi G} (R_h - 2\Lambda) \right] |\Psi[h]\rangle = 0, \quad 2D_b h_{ac} M_{de}^{bc} \frac{\delta}{\delta h_{de}} |\Psi[h]\rangle = 0. \quad (67)$$

The constraints (67) restrict the space of states, which are dynamically possible,  $|\Psi[h(x)]\rangle$ , to a subspace  $\mathcal{V}_{dyn}$  of the space of all states  $\mathcal{V}$ :  $\mathcal{V}_{dyn} \subset \mathcal{V}$ . The problems concerning the definition of an inner product in quantum geometrodynamics remain the same as in the usual case. But of course, the quaternionic quantization principle can analogously be transferred to the new formulation of Hamiltonian general relativity given in [98],[99], on which loop quantum gravity is based, which has been developed in [100],[101]. This is done in the next subsection.

### B. Quaternionic Quantization of Ashtekars Variables

The Hamiltonian formulation of general relativity based on Ashtekars variables contains the connection of the gravitational field on the submanifold  $\Sigma$  as the decisive quantity, which is expressed as special spin connection. The canonical conjugated quantity is the tetrad field on  $\Sigma$  multiplied with the square root of the three metric. Concretely, the Ashtekar variables are defined as follows:

$$A_a^i = \frac{\Gamma_a^i + \beta K_a^i}{G}, \quad E_i^a = \sqrt{h} e_i^a, \quad (68)$$

where  $K_a^i$  denotes the extrinsic curvature defined in (50) and  $\Gamma_a^i$  is defined as follows:  $\Gamma_a^i = -\frac{1}{2}\omega_{ajk}\epsilon^{ijk}$ , with  $\omega_{ajk}$  describing the spin connection.  $\beta$  denotes the Immirzi parameter. Since the connection  $A_a^i$  and the canonical conjugated variable  $E_i^a$  contain only one space-time index, the quaternionic quantization principle can directly be transferred from quantum mechanics to general relativity by using the quaternionic quantization tensor  $\alpha^\mu_\nu$  with respect to its spatial part referring to  $\Sigma$ , which shall be called  $P_b^a$ ,

$$P_b^a = \begin{pmatrix} i & \varkappa j & \varkappa j \\ \varkappa j & i & \varkappa j \\ \varkappa j & \varkappa j & i \end{pmatrix}, \quad (69)$$

and postulating the following generalization of the commutation relation:

$$[\hat{A}_a^i(x), \hat{E}_j^b(y)] = 8\pi\beta i \delta_a^b \delta_j^i \delta(x-y) \quad \longrightarrow \quad [\hat{A}_a^i(x), \hat{E}_j^b(y)] = 8\pi\beta P_b^a \delta_j^i \delta(x-y). \quad (70)$$

The quantization principle (70) leads to the following representation of the operators with respect to the connection,

$$\hat{A}_a^i(x) |\Psi[A(x)]\rangle = A_a^i(x) |\Psi[A(x)]\rangle, \quad \hat{E}_i^a(x) |\Psi[A(x)]\rangle = -8\pi\beta P_b^a \frac{\delta}{\delta A_b^i(x)} |\Psi[A(x)]\rangle. \quad (71)$$

The inverse of the three dimensional quaternionic quantization tensor  $P_b^a$  reads:

$$(P^{-1})_b^a = \begin{pmatrix} u_i i + u_j j & v_i i + v_j j & v_i i + v_j j \\ v_i i + v_j j & u_i i + u_j j & v_i i + v_j j \\ v_i i + v_j j & v_i i + v_j j & u_i i + u_j j \end{pmatrix}, \quad (72)$$

where the coefficients of the quaternionic entries of  $(P^{-1})^a_b$  are of the following shape:

$$u_i = -\frac{3\kappa^2 + 1}{4\kappa^4 + 5\kappa^2 + 1}, \quad u_j = \frac{2\kappa^3}{4\kappa^4 + 5\kappa^2 + 1}, \quad v_i = \frac{\kappa^2}{4\kappa^4 + 5\kappa^2 + 1}, \quad v_j = -\frac{2\kappa^3 + \kappa}{4\kappa^4 + 5\kappa^2 + 1}. \quad (73)$$

Of course, analogue to the case of  $\alpha^\mu_\nu$  as well as  $M_{cd}^{ab}$ , also the components  $P^a_b$  fulfil nontrivial commutation relations,

$$[P^{ab}, P^{cd}] = \Xi^{abcd}, \quad (74)$$

where  $\Xi_{abcd}$  is a tensor of fourth order, which can be written as a matrix,

$$\Xi^{abcd} = \kappa \begin{pmatrix} \xi_i^{ab} & \xi_j^{ab} & \xi_j^{ab} \\ \xi_j^{ab} & \xi_i^{ab} & \xi_j^{ab} \\ \xi_j^{ab} & \xi_j^{ab} & \xi_i^{ab} \end{pmatrix}, \quad (75)$$

which contains tensors of second order,  $\xi_i^{ab}$  and  $\xi_j^{ab}$ , which can be written as matrices again,

$$\xi_i^{ab} = \begin{pmatrix} 0 & 2k & 2k \\ 2k & 0 & 2k \\ 2k & 2k & 0 \end{pmatrix}, \quad \xi_j^{ab} = \begin{pmatrix} -2k & 0 & 0 \\ 0 & -2k & 0 \\ 0 & 0 & -2k \end{pmatrix}. \quad (76)$$

This leads of course also in this formulation to nontrivial commutation relations of the components of the operator of the connection with each other as well as between the components the operator of the conjugated variable with each other, which are analogue to (65) and (66) and are given by

$$[\hat{A}_a^i(x), \hat{A}_b^j(y)] = \Xi_{abcd} (P^{-1})^{ce} (P^{-1})^{df} \hat{A}_e^i(x) \hat{A}_f^j(y), \quad (77)$$

and

$$[\hat{E}_i^a(x), \hat{E}_j^b(y)] = \Xi^{abcd} (P^{-1})_{ce} (P^{-1})_{df} \hat{E}_i^e(x) \hat{E}_j^f(y). \quad (78)$$

If the special choice  $\beta = i$  is made, the Hamiltonian constraint is given by

$$\hat{\mathcal{H}}_\tau |\Psi[A]\rangle = \epsilon^{ijk} \hat{F}_{kab} \hat{E}_i^a \hat{E}_j^b |\Psi[A]\rangle = 16\pi^2 \beta^2 \epsilon^{ijk} F_{kab} P_c^a P_d^b \frac{\delta}{\delta A_c^i} \frac{\delta}{\delta A_d^j} |\Psi[A]\rangle = 0, \quad (79)$$

and the diffeomorphism constraint is given by

$$\hat{\mathcal{H}}_a |\Psi[A]\rangle = \hat{F}_{ab}^i \hat{E}_i^b |\Psi[A]\rangle = -8\pi\beta F_{ab}^i P_c^b \frac{\delta}{\delta A_c^i} |\Psi[A]\rangle = 0, \quad (80)$$

where the field strength tensor  $F_{ab}^i$  is defined as

$$F_{ab}^i = 2G\partial_a A_b^i + G^2 \epsilon_{jk}^i A_a^j E_b^k. \quad (81)$$

In case of the Ashtekar formulation, the Gauss constraint additionally appears, which in case of the postulation of the quaternionic quantization principle reads

$$\mathcal{D}_a \hat{E}_i^a |\Psi[A]\rangle = \mathcal{D}_a P_b^a \frac{\delta}{\delta A_b^i} |\Psi[A]\rangle = 0, \quad \text{with} \quad \mathcal{D}_a E_i^a = \partial_a E_i^a + G\epsilon_{ijk} A_a^j E_a^{ka}. \quad (82)$$

As approach for the formulation of an inner product,  $\langle \cdot | \cdot \rangle$ , in analogy to the usual case one can postulate,

$$\langle \Phi | \Psi \rangle = \int_{\mathcal{V}_{dyn}} \mathcal{D}\mu[A] \Phi^*[A] \Psi[A], \quad (83)$$

which constitutes a Hilbert space  $\mathcal{H}_G$  based on  $\mathcal{V}_{dyn}$ .  $\Psi^*[A]$  still denotes the quaternionic conjugated quantity to  $\Psi[A]$ . The operators  $\hat{A}_a^i$  and  $\hat{E}_i^a$ , which can be represented in the connection representation according to (71), are self-adjoint with respect to the inner product (83).

## VI. QUATERNIONIC QUANTIZATION OF SUPERGRAVITY

### A. The Quaternionic Quantization Principle in case of the Appearance of Dirac Brackets

The quaternionic quantization principle has already been considered for the special cases of quantum mechanics and usual canonical general relativity in the last sections. In this section the quaternionic quantization procedure shall be transferred to an extension of general relativity,  $\mathcal{N} = 1$  supergravity namely. Of course, the canonical formulation of supergravity has to be considered. The canonical quantization of supergravity has first been developed in [102],[103]. Further developments can for example be found in [104],[105],[106],[107],[108],[109],[110],[111],[112],[113],[114],[115],[116],[117]. The quantization procedure becomes more intricate in the case of supergravity, since because of the appearance of second class constraints, the quantization is not performed by postulating the commutation relations being equal to  $i$  times the corresponding classical Poisson-brackets, but the Poisson brackets have to be replaced by Dirac brackets. This means that some commutators are proportional to more complicated tensors than just a product of delta functions and nontrivial commutators between the components of the operators of the variables with each other can already arise in the usual case. Thus the question arises how the quaternionic quantization principle has to be implemented, if Dirac brackets appear. Accordingly the quaternionic quantization principle has to be formulated in a more general form and it is necessary to find a general setting of the quaternionic quantization principle as generalization of general quantum theory. In quantum mechanics the transition from the usual quantization principle to the quaternionic quantization principle is performed by replacing the Kronecker symbol  $\delta^\mu_\nu$  by the quaternionic quantization tensor  $\alpha^\mu_\nu$ , what has been done in (7). For an arbitrary vector  $A_a$  and its corresponding canonical conjugated quantity  $B_b$ , which are quantized by transferring Poisson brackets to commutators, this means

$$[\hat{A}_a, \hat{B}_b] = i\delta_{ab} \quad \longrightarrow \quad [\hat{A}_a, \hat{B}_b] = P_{ab}^D, \quad (84)$$

where  $P_{ab}^D$  is the  $D$ -dimensional generalization of  $P_{ab}$ . It has now to be formulated the generalization of (84) to enable the transfer to theories, where second class constraints appear and thus Dirac brackets have to be considered meaning that the commutator is postulated to be equal to a more complicated tensor than the Kronecker symbol already in the usual case. To obtain such a formulation, it is helpful to reexpress  $P_{ab}^D$  as follows:

$$P_{ab}^D = i\mathcal{Q}_{ab}^{cd}\delta_{cd}, \quad (85)$$

where  $\mathcal{Q}_{ab}^{cd}$  is a quaternionic tensor of fourth order, which reads

$$\mathcal{Q}_{ab}^{cd} = \begin{pmatrix} 1 & -\kappa k & \dots & \dots & -\kappa k \\ -\kappa k & 1 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 1 & -\kappa k \\ -\kappa k & \dots & \dots & -\kappa k & 1 \end{pmatrix} \otimes \frac{1}{D} \begin{pmatrix} 1 & i & \dots & \dots & i \\ i & 1 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 1 & i \\ i & \dots & \dots & i & 1 \end{pmatrix} \equiv \mathcal{P}_{ab} \otimes \frac{\mathcal{N}^{cd}}{D}, \quad a, b, c, d = 1 \dots D, \quad (86)$$

where  $D$  denotes the number of dimensions of the space the quantities live in, which have to be quantized, and  $\mathcal{N}^{ab}$  has been chosen in such a way that  $\mathcal{N}^{ab}\delta_{ab} = D$ , that it is invertible and its non diagonal elements do not vanish anyhow, what is important with respect to the Dirac brackets. In the special case  $D = 3$  it holds  $i\mathcal{P}_{ab} = P_{ab}$ . Remember that  $P_{ab}$  has been defined in (69). This means that the tensor  $\mathcal{Q}_{ab}^{cd}$  serves as a mediation tensor between the usual value of the commutator and the generalized value of the commutator. Accordingly it is possible to generalize this quantization principle by transforming any tensor on the right hand side of the usual quantization principle with the transformation tensor  $\mathcal{Q}_{ab}^{cd}$ . This means for the quantization of an arbitrary vector  $A_a$  and its corresponding canonical conjugated quantity  $B_b$ , which are usually quantized by transferring the Dirac bracket to a commutator,

$$[\hat{A}_a, \hat{B}_b]_{+/-} = i\{A_a, B_b\}_D \quad \longrightarrow \quad [\hat{A}_a, \hat{B}_b]_{+/-} = i\mathcal{Q}_{ab}^{cd}\{A_c, B_d\}_D, \quad (87)$$

where  $\{\cdot, \cdot\}_D$  denotes the corresponding Dirac bracket being defined as

$$\{A, B\}_D = \{A, B\}_P - \{A, \Theta_i\}_P (\Omega^{-1})^{ij} \{\Theta_j, B\}_P, \quad (88)$$

where  $\{\cdot, \cdot\}_P$  denotes the usual Poisson bracket, the  $\Theta_i$  denote the second class constraints and  $\Omega_{ij} = \{\Theta_i, \Theta_j\}_P$ . In case of  $D = 3$  as it appears with respect to the generalization of the quantization of  $\mathcal{N} = 1$  supergravity in usual (3+1)-dimensional space-time, the quaternionic mediation tensor  $\mathcal{Q}_{ab}^{cd}$  multiplied with  $i$ , which has to be applied to the value of the Dirac bracket to obtain the generalized commutator and is denoted by  $Q_{ab}^{cd}$ , can be written as follows:

$$Q_{ab}^{cd} = i\mathcal{Q}_{ab}^{cd} = i\mathcal{P}_{ab} \otimes \frac{\mathcal{N}^{cd}}{3} = P_{ab} \otimes \frac{\mathcal{N}^{cd}}{3}. \quad (89)$$

Concerning the formulation of the generalized quantization rules arising from the Dirac brackets, the inverse of  $Q_{ab}^{cd}$  will have to be used, which can be written as

$$(Q^{-1})_{cd}^{ab} = -(\mathcal{Q}^{-1})_{cd}^{ab} i = -3(\mathcal{P}^{-1})^{ab} \otimes (\mathcal{N}^{-1})_{cd} i = 3(P^{-1})^{ab} \otimes (\mathcal{N}^{-1})_{cd}. \quad (90)$$

$(P^{-1})^{ab}$  has been calculated in (72). The commutator of the several components of  $Q_{ab}^{cd}$  with each other reads:

$$[Q_{ab}^{cd}, Q_{ef}^{gh}] = \Xi_{abef} \frac{\mathcal{N}^{cd} \mathcal{N}^{gh}}{9} \equiv \Gamma_{abef}^{cdgh} + \frac{P_{ef}}{9} \Delta_{ab}^{gh} \mathcal{N}^{cd} - \frac{P_{ab}}{9} \Delta_{ef}^{cd} \mathcal{N}^{gh}, \quad (91)$$

where the last equalization serves as a definition of  $\Gamma_{abef}^{cdgh}$ ,  $\Xi_{abef}$  has already been determined in (75) and  $\Delta_{ab}^{cd}$  can be written by using the definition (76) and thus reads as follows:

$$\Delta_{ab}^{cd} = \begin{pmatrix} 0 & -\xi_{i \ ab} & -\xi_{i \ ab} \\ -\xi_{i \ ab} & 0 & -\xi_{i \ ab} \\ -\xi_{i \ ab} & -\xi_{i \ ab} & 0 \end{pmatrix}. \quad (92)$$

## B. Quaternionic Quantization Principle in $\mathcal{N} = 1$ Supergravity

To formulate supergravity, the tetrad formalism has to be used. The tetrad field is related to the metric field as usual by the following relation:  $g_{\mu\nu} = \eta_{mn} e_\mu^m e_\nu^n$ , where  $\eta_{mn}$  denotes the Minkowski metric. Concerning the canonical quantization as well as its quaternionic extension, it is helpful to use the spinor representation of Minkowski vectors given in [118],[119] and [120] for example, which is obtained by using the Pauli matrices. This means that the tetrad field  $e_\mu^m$  can be rewritten as

$$e_\mu^{AA'} = e_\mu^n \bar{\sigma}_n^{AA'}. \quad (93)$$

Here the matrices  $\bar{\sigma}_n^{AA'}$  are considered as components of a four-vector containing the Pauli matrices and the negative unity matrix,

$$\bar{\sigma}_0 = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{\sigma}^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \bar{\sigma}^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \bar{\sigma}^3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (94)$$

Indices of the spinor representation are raised and lowered by the tensor  $\epsilon^{AB}$ , which is the total antisymmetric tensor with respect to the spinor space. The new quantity  $n^{AA'}$  directly related to the unit normal  $n^\mu$  to the three dimensional submanifold  $\Sigma$  on which the three metric lives is defined as follows:

$$n^{AA'} = e_\mu^{AA'} n^\mu, \quad (95)$$

and fulfils the following relations:

$$n_{AA'} e_a^{AA'} = 0, \quad n_{AA'} n^{AA'} = 1. \quad (96)$$

The unit normal, the components of the tetrad field and  $N$  as well as  $N^a$  are further related by



$$e_0^{AA'} = N n^{AA'} + N^a e_a^{AA'}, \quad (97)$$

and the action of  $\mathcal{N} = 1$  supergravity is given by

$$\mathcal{S}_{\mathcal{N}=1} = \frac{1}{16\pi G} \int d^4x \, e \, e_a^\mu e_b^\nu R_{\mu\nu}^{ab} + \frac{1}{2} \int d^4x \, e \, \epsilon^{\mu\nu\rho\sigma} \left( \bar{\psi}_\mu^{A'} e_{AA'\nu} D_\rho \psi_\sigma^A + h.c. \right), \quad (98)$$

where  $e = \det(e_\mu^m)$ ,  $\epsilon^{\mu\nu\rho\sigma}$  denotes the total antisymmetric tensor in four dimensions and the covariant derivative applied to a spinor reads

$$D_\mu \psi_\nu^A = \partial_\mu \psi_\nu^A + \omega_{\mu B}^A \psi_\nu^B, \quad (99)$$

where  $\omega_{\mu B}^A$  denotes the spin connection in the spinor representation. Because of the appearance of the Rarita Schwinger field the existence of a torsion is induced,

$$\mathcal{T}_{\mu\nu}^{AA'} = D_{[\mu} e_{\nu]}^{AA'} = -4\pi i \bar{\psi}_{[\mu}^{A'} \psi_{\nu]}^A, \quad (100)$$

which is however assumed to vanish in the further consideration of this paper. The action of  $\mathcal{N} = 1$  supergravity (98) has three symmetries. Since it is a supersymmetric theory, it of course contains a supersymmetry and is invariant under the following corresponding transformations:

$$\delta e_\mu^{AA'} = -i\sqrt{8\pi G} \left( \zeta^A \bar{\psi}_\mu^{A'} + \bar{\zeta}^{A'} \psi_\mu^A \right), \quad \delta \psi_\mu^A = \frac{D_\mu \zeta^A}{\sqrt{2\pi G}}, \quad \delta \bar{\psi}_\mu^{A'} = \frac{D_\mu \bar{\zeta}^{A'}}{\sqrt{2\pi G}}, \quad (101)$$

where  $\zeta^A$  and  $\bar{\zeta}^{A'}$  denote the transformation parameter of the supersymmetry and its adjoint. Besides it contains local Lorentz symmetry and accordingly is invariant under the following transformations:

$$\delta e_\mu^{AA'} = L_B^A e_\mu^{BA'} + \bar{L}_{B'}^{A'} e^{AB'\mu}, \quad \delta \psi_\mu^A = L_B^A \psi_\mu^B, \quad \delta \bar{\psi}_\mu^{A'} = \bar{L}_{B'}^{A'} \bar{\psi}_\mu^{B'}, \quad (102)$$

where  $L_B^A$  and  $\bar{L}_{B'}^{A'}$  denote the transformation parameter of the Lorentz group and its adjoint, and it contains local symmetry under coordinate transformations and thus contains diffeomorphism invariance, which means that it is invariant under the following transformations:

$$\delta e_\mu^{AA'} = z^\nu \partial_\nu e_\mu^{AA'} + e_\nu^{AA'} \partial_\mu z^\nu, \quad \delta \psi_\mu^A = z^\nu \partial_\nu \psi_\mu^A + \psi_\nu^A \partial_\mu z^\nu, \quad (103)$$

where  $z^\mu$  denotes the corresponding local translation parameter. The canonical conjugated variables to the tetrad field  $e_\mu^{AA'}$  as well as the Rarita Schwinger field  $\psi_a^A$  and its adjoint  $\bar{\psi}_a^{A'}$  are as usual defined with respect to the the action of  $\mathcal{N} = 1$  supergravity (98) and accordingly the conjugated variable to the tetrad field  $p_{AA'}^a$  is given by

$$p_{AA'}^a = \frac{\delta \mathcal{S}_{\mathcal{N}=1}}{\delta \dot{e}_a^{AA'}}. \quad (104)$$

$p_{AA'}^a$  is related to the canonical conjugated variable of  $h_{ab}$  within the usual formulation of canonical general relativity used in quantum geometrodynamics by

$$\pi^{ab} = \frac{e^{AA'a} p_{AA'}^b + e^{AA'b} p_{AA'}^a}{2}, \quad (105)$$

and the canonical conjugated variables to  $\psi_a^A$  and  $\bar{\psi}_a^{A'}$  read as follows:

$$(\pi_\psi)_A^a = \frac{\delta \mathcal{S}_{\mathcal{N}=1}}{\delta \dot{\psi}_a^A} = -\frac{1}{2} \epsilon^{abc} \bar{\psi}_b^{A'} e_{AA'c}, \quad (\tilde{\pi}_{\bar{\psi}})_{A'}^a = \frac{\delta \mathcal{S}_{\mathcal{N}=1}}{\delta \dot{\bar{\psi}}_a^{A'}} = \frac{1}{2} \epsilon^{abc} \psi_b^A e_{AA'c}. \quad (106)$$

The complete Hamiltonian of  $\mathcal{N} = 1$  supergravity corresponding to the action (98) reads

$$H = \int d^3x \left( N\mathcal{H}_\tau + N^a \mathcal{H}_a + \psi_0^A S_A + \bar{S}_{A'} \bar{\psi}_0^{A'} - \omega_{AB0} J^{AB} - \bar{\omega}_{A'B'0} \bar{J}^{A'B'} \right), \quad (107)$$

which corresponds to the following constraints:

$$J_{AB} = 0, \quad \bar{J}_{A'B'} = 0, \quad \mathcal{H} = 0, \quad \mathcal{H}_a = 0, \quad S_A = 0, \quad \bar{S}_{A'} = 0, \quad (108)$$

if it is varied with respect to  $N$ ,  $N^a$ ,  $\psi_0^A$ ,  $\bar{\psi}_0^{A'}$ ,  $\omega_{AB0}$ ,  $\bar{\omega}_{A'B'0}$ . The part of the Hamiltonian density  $\mathcal{H}_\tau$  of (107) belonging to the time direction reads under the condition that the torsion vanishes

$$\mathcal{H}_\tau = 16\pi G G_{abcd} \pi^{ab} \pi^{cd} - \frac{\sqrt{h}R}{16\pi G} + \left( \frac{1}{2} \epsilon^{abc} \bar{\psi}_a^{A'} n_{AA'} \mathcal{D}_b \psi_c^A + h.c. \right), \quad (109)$$

and the part of the Hamiltonian density  $\mathcal{H}_a$  belonging to the spacelike directions with the submanifold  $\Sigma$  then reads

$$\mathcal{H}_a = -2h_{ab} \mathcal{D}_c \pi^{bc} + \left( \frac{1}{2} \epsilon^{bcd} \bar{\psi}_b^{A'} e_{AA'a} \mathcal{D}_c \psi_d^A + h.c. \right). \quad (110)$$

The expression  $S_A$  and its conjugated quantity  $\bar{S}_{A'}$  represent the generators of supersymmetry transformations and are defined as

$$S_A = \epsilon^{abc} e_{AA'a} \mathcal{D}_b \bar{\psi}_c^{A'} - i4\pi G p_{AA'}^a \bar{\psi}_a^{A'}, \quad \bar{S}_{A'} = \epsilon^{abc} e_{AA'a} \mathcal{D}_b \psi_c^A + i4\pi G p_{AA'}^a \psi_a^A. \quad (111)$$

The expression  $J_{AB}$  and its conjugated quantity are the generators of Lorentz transformations and are defined as

$$\begin{aligned} J_{AB} &= e_{(A}^{A'} p_{B)A'a} + \psi_{(A}^a (\pi_\psi)_{B)a} = e_{(A}^{A'} p_{B)A'a} - \frac{1}{2} \psi_{(A}^a \epsilon_{abc} \bar{\psi}^{A'b} e_{B)a}^c, \\ \bar{J}_{A'B'} &= e_{(A'}^{Aa} p_{B')Aa} + \bar{\psi}_{(A'}^a (\tilde{\pi}_{\bar{\psi}})_{B')a} = e_{(A'}^{Aa} p_{B')Aa} + \frac{1}{2} \bar{\psi}_{(A'}^a \epsilon_{abc} \psi^{Ab} e_{B')a}^c. \end{aligned} \quad (112)$$

As already mentioned at the beginning of the first subsection of this section, the appearance of second class constraints in supergravity implies that one has to introduce Dirac brackets, which replace the usual Poisson brackets concerning the formulation of the quantization rules. The general definition of a Dirac bracket is given in (88) and in case of supergravity the Dirac brackets between the several quantities read

$$\begin{aligned} \left\{ e_a^{AA'}(x), e_b^{BB'}(y) \right\}_D &= 0, \\ \left\{ e_a^{AA'}(x), p_{BB'}^b(y) \right\}_D &= \epsilon_{AB}^A \epsilon_{B'}^{A'} \delta_a^b \delta(x-y), \\ \left\{ p_{AA'}^a(x), p_{BB'}^b(y) \right\}_D &= \frac{1}{4} \epsilon^{bcd} \psi_{Bd} D_{AB'ec} \epsilon^{aef} \bar{\psi}_{A'f} \delta(x-y) + h.c., \\ \left\{ \psi_a^A(x), \psi_b^B(y) \right\}_D &= 0, \\ \left\{ \psi_a^A(x), \bar{\psi}_b^{A'}(y) \right\}_D &= -D_{ab}^{AA'} \delta(x-y), \\ \left\{ e_a^{AA'}(x), \psi_b^B(y) \right\}_D &= 0, \\ \left\{ p_{AA'}^a(x), \psi_b^B(y) \right\}_D &= \frac{1}{2} \epsilon^{acd} \psi_{Ad} D_{A'b c}^B \delta(x-y), \end{aligned} \quad (113)$$

where the quantity  $D_{ab}^{AA'}$  has been defined according to

$$D_{ab}^{AA'} = -\frac{2i}{\sqrt{h}} e_b^{AB'} e_{BB'a} n^{BA'}. \quad (114)$$

The generators of supersymmetry (111) fulfil the following Dirac brackets:

$$\{S_A(x), S_B(y)\}_D = 0, \quad \{\bar{S}_{A'}(x), \bar{S}_{B'}(y)\}_D = 0, \quad \{S_A(x), \bar{S}_{A'}(y)\}_D = i4\pi G\mathcal{H}_{AA'}(x)\delta(x-y). \quad (115)$$

In the usual quantization procedure, the Dirac brackets (113) have to be converted to commutators multiplied with  $-i$ . A quaternionic quantization is performed by referring to (87) and accordingly the commutator has to be extended by applying the operator  $\mathcal{Q}_{ab}^{cd}$  to it. But this holds only for the commutation relations involving the components of the quantized variables and their corresponding canonical conjugated quantities and not for the other commutation relations, which have to be derived from them because of their dependence on them. The choice, if commutation or anticommutation relations have to be used, is completely analogue to the usual case and this means that commutation relations are assumed, if at least one of the variables is Grassmann-even and otherwise anticommutation relations are assumed. This means that the following commutation and anticommutation relations are postulated:

$$\left[e_a^{AA'}(x), p_{BB'}^b(y)\right]_- = i\delta_{ed}\delta^{bf}\mathcal{Q}_{af}^{ce}\epsilon_B^A\epsilon_{B'}^{A'}\delta_c^d\delta(x-y), \quad \left[\psi_a^A(x), \bar{\psi}_b^{A'}(y)\right]_+ = -i\mathcal{Q}_{ab}^{cd}D_{cd}^{AA'}\delta(x-y). \quad (116)$$

Besides one is led to the following commutation relation, if one assumes the quantization principle (87) also to be valid with respect to the commutation relations between the components of  $p_{AA'}^a(x)$  and  $\psi_a^A(x)$  as commutation relation between a field operator and the canonical conjugated field operator belonging to another field operator,

$$\left[p_{AA'}^a(x), \psi_b^B(y)\right]_- = \frac{i}{2}\delta_{gf}\delta^{ae}\mathcal{Q}_{eb}^{fh}\epsilon^{gcd}\psi_{Ad}D_{A'hc}^B\delta(x-y). \quad (117)$$

Since the transition rule (87) to obtain the commutation relations of the corresponding quantum theory, if the Dirac brackets of the classical theory are given, holds only with respect to the commutation relations between the components of a field operator and the components of a canonical conjugated variable because the other commutation relations depend on them, the Dirac brackets given in (113) accordingly have to be converted to commutation relations by deriving them from the properties of the operators  $\hat{e}_a^{AA'}$ ,  $\hat{p}_{AA'}^a$ ,  $\hat{\psi}_a^{AA'}$  and  $\hat{\bar{\psi}}_a^{AA'}$ . These operators are already defined by the commutation relations (116) and (117). The operators  $\hat{e}_a^{AA'}$ ,  $\hat{p}_{AA'}^a$ ,  $\hat{\psi}_a^{AA'}$  and  $\hat{\bar{\psi}}_a^{AA'}$  fulfilling the commutation relations (116) and (117) can be represented by referring on the corresponding states depending on  $e_a^{AA'}$  and  $\psi_a^{AA'}$ ,  $|\Psi\rangle = |\Psi[e, \psi]\rangle$ , and in this representation the operators  $\hat{\psi}_a^{AA'}$  and  $\hat{\bar{\psi}}_a^{AA'}$  are of the following shape:

$$\hat{\psi}_a^A(x)|\Psi[e(x), \psi(x)]\rangle = \psi_a^A(x)|\Psi[e(x), \psi(x)]\rangle, \quad \hat{\bar{\psi}}_a^{A'}(x)|\Psi[e(x), \psi(x)]\rangle = -i\mathcal{Q}_{ba}^{cd}D_{cd}^{AA'}\frac{\delta}{\delta\psi_b^A(x)}|\Psi[e(x), \psi(x)]\rangle, \quad (118)$$

and the operators  $\hat{e}_a^{AA'}$  and  $\hat{p}_{AA'}^a$  look as follows:

$$\begin{aligned} \hat{e}_a^{AA'}(x)|\Psi[e(x), \psi(x)]\rangle &= e_a^{AA'}(x)|\Psi[e(x), \psi(x)]\rangle, \\ \hat{p}_{AA'}^a(x)|\Psi[e(x), \psi(x)]\rangle &= \left[-i\delta_{ed}\delta^{af}\mathcal{Q}_{bf}^{ce}\delta_c^d\frac{\delta}{\delta e_b^{AA'}(x)} - \frac{i}{2}\delta_{gf}\delta^{ae}\mathcal{Q}_{eb}^{fh}\epsilon^{gcd}\psi_{Ad}D_{A'hc}^B\frac{\delta}{\delta\psi_b^B(x)}\right]|\Psi[e(x), \psi(x)]\rangle. \end{aligned} \quad (119)$$

If the states are assumed to depend on  $\hat{p}_{AA'}^a$  and  $\hat{\bar{\psi}}_a^{AA'}$ ,  $|\Psi\rangle = |\Psi[p(x), \bar{\psi}(x)]\rangle$ , then the operators  $\hat{\psi}_a^{AA'}$  and  $\hat{\bar{\psi}}_a^{AA'}$  can be represented as

$$\hat{\psi}_a^A(x)|\Psi[p(x), \bar{\psi}(x)]\rangle = -i\mathcal{Q}_{ba}^{cd}D_{cd}^{AA'}\frac{\delta}{\delta\bar{\psi}_b^A(x)}|\Psi[p(x), \bar{\psi}(x)]\rangle, \quad \hat{\bar{\psi}}_a^{A'}(x)|\Psi[p(x), \bar{\psi}(x)]\rangle = \bar{\psi}_a^{A'}(x)|\Psi[p(x), \bar{\psi}(x)]\rangle, \quad (120)$$

and the operators  $\hat{e}_a^{AA'}$  and  $\hat{p}_{AA'}^a$  look as follows:

$$\begin{aligned} \hat{e}_a^{AA'}|\Psi[p(x), \bar{\psi}(x)]\rangle &= i\delta_{ed}\delta^{bf}\mathcal{Q}_{af}^{ce}\delta_c^d\frac{\delta}{\delta p_{AA'}^b(x)}|\Psi[p(x), \bar{\psi}(x)]\rangle, \\ \hat{p}_{AA'}^a(x)|\Psi[p(x), \bar{\psi}(x)]\rangle &= p_{AA'}^a(x)|\Psi[p(x), \bar{\psi}(x)]\rangle. \end{aligned} \quad (121)$$

The other commutation relations representing no independent assumptions can now be derived from the operators (118), (120), (120) and (121), which are already defined through the commutation relations considered in (116) and (117). Accordingly the complete set of commutation relations reads as follows:

$$\begin{aligned}
[e_a^{AA'}(x), e_g^{GG'}(y)]_- &= \Gamma_{afgl}^{ceik} \delta_{ed} \delta^{bf} \delta_c^d \delta_{kj} \delta^{hl} \delta_i^j \left[ \delta_{bm} \delta^{np} (\mathcal{Q}^{-1})_{np}^{mq} i e_q^{AA'}(x) \right] \left[ \delta_{hr} \delta^{st} (\mathcal{Q}^{-1})_{st}^{ru} i e_u^{GG'}(y) \right], \\
[e_a^{AA'}(x), p_{BB'}^b(y)]_- &= i \delta_{ed} \delta^{bf} \mathcal{Q}_{af}^{ce} \epsilon_B^A \epsilon_{B'}^{A'} \delta_c^d \delta(x-y), \\
[p_{AA'}^a(x), p_{II'}^i(y)]_- &= \Gamma_{bfjn}^{cekm} \delta_{ed} \delta^{af} \delta_c^d \delta_{ml} \delta^{in} \delta_k^l \\
&\quad \times \left\{ \delta^{uv} (\mathcal{Q}^{-1})_{uv}^{wb} \left[ i p_{AA'w}(x) - \frac{1}{2} \delta_{sr} \mathcal{Q}_{wo}^{st} \epsilon^{rqp} \psi_{Aq}(x) D_{A'tp}^B(x) \left( \mathcal{C}_{BC'}^{xy}(x) (\mathcal{Q}^{-1})_{xy}^{oz} i \bar{\psi}_z^{C'}(x) \right) \right] \right\} \\
&\quad \times \left\{ \delta^{\bar{u}\bar{v}} (\mathcal{Q}^{-1})_{\bar{u}\bar{v}}^{\bar{w}\bar{j}} \left[ i p_{II'\bar{w}}(y) - \frac{1}{2} \delta_{\bar{s}\bar{r}} \mathcal{Q}_{\bar{w}\bar{o}}^{\bar{s}\bar{t}} \epsilon^{\bar{r}\bar{q}\bar{p}} \psi_{I\bar{q}}(y) D_{I'\bar{t}\bar{p}}^D(y) \left( \mathcal{C}_{DE'}^{\bar{x}\bar{y}}(y) (\mathcal{Q}^{-1})_{\bar{x}\bar{y}}^{\bar{o}\bar{z}} i \bar{\psi}_{\bar{z}}^{E'}(y) \right) \right] \right\} \\
&\quad + \frac{1}{2} \Gamma_{bfmj}^{cenq} \delta_{ed} \delta^{af} \delta_c^d \delta_{pn} \delta^{im} \epsilon^{plk} \psi_{Il}(y) D_{I'qk}^J(y) \left\{ \delta^{\bar{h}\bar{i}} (\mathcal{Q}^{-1})_{\bar{h}\bar{i}}^{\bar{j}\bar{b}} \left[ i p_{AA'\bar{j}}(x) \right. \right. \\
&\quad \left. \left. - \frac{1}{2} \delta_{\bar{f}\bar{e}} \mathcal{Q}_{\bar{j}\bar{b}}^{\bar{f}\bar{g}} \epsilon^{\bar{e}\bar{d}\bar{c}} \psi_{A\bar{d}}(x) D_{A'\bar{g}\bar{c}}^B(x) \left( \mathcal{C}_{BC'}^{\bar{k}\bar{l}}(x) (\mathcal{Q}^{-1})_{\bar{k}\bar{l}}^{\bar{b}\bar{m}} i \bar{\psi}_{\bar{m}}^{C'}(x) \right) \right] \right\} \left[ \mathcal{C}_{JB'}^{yz}(y) (\mathcal{Q}^{-1})_{yz}^{jx} i \bar{\psi}_x^{B'}(y) \right] \\
&\quad + \frac{1}{2} \Gamma_{ebjn}^{fhkm} \delta_{gf} \delta^{ae} \epsilon^{gcd} \delta_{ml} \delta^{in} \delta_k^l \psi_{Ad}(x) D_{A'hc}^B(x) \left[ \mathcal{C}_{BB'}^{yz}(x) (\mathcal{Q}^{-1})_{yz}^{bx} i \bar{\psi}_x^{B'}(x) \right] \\
&\quad \times \left\{ \delta^{\bar{h}\bar{i}} (\mathcal{Q}^{-1})_{\bar{h}\bar{i}}^{\bar{j}\bar{j}} \left[ i p_{II'\bar{j}}(y) - \frac{1}{2} \delta_{\bar{f}\bar{e}} \mathcal{Q}_{\bar{j}\bar{b}}^{\bar{f}\bar{g}} \epsilon^{\bar{e}\bar{d}\bar{c}} \psi_{A\bar{d}}(y) D_{A'\bar{g}\bar{c}}^C(y) \left( \mathcal{C}_{CC'}^{\bar{k}\bar{l}}(y) (\mathcal{Q}^{-1})_{\bar{k}\bar{l}}^{\bar{b}\bar{m}} i \bar{\psi}_{\bar{m}}^{C'}(y) \right) \right] \right\} \\
&\quad + \frac{1}{4} \Gamma_{ebmj}^{fhmq} \delta_{gf} \delta^{ae} \epsilon^{gcd} \delta_{pn} \delta^{im} \epsilon^{plk} \psi_{Ad}(x) D_{A'hc}^B(x) \psi_{Il}(y) D_{I'qk}^J(y) \\
&\quad \times \left[ \mathcal{C}_{BB'}^{vw}(x) (\mathcal{Q}^{-1})_{vw}^{bu} i \bar{\psi}_u^{B'}(x) \right] \left[ \mathcal{C}_{JC'}^{yz}(y) (\mathcal{Q}^{-1})_{yz}^{jx} i \bar{\psi}_x^{C'}(y) \right] \\
&\quad + \frac{1}{4} i \mathcal{Q}_{eb}^{fh} i \mathcal{Q}_{mj}^{nq} \delta_{gf} \delta^{ae} \epsilon^{gcd} \delta_{pn} \delta^{im} \epsilon^{plk} \left[ \psi_{Ad}(x) D_{A'hc}^B(x), \psi_{Il}(y) D_{I'qk}^J(y) \right] \\
&\quad \times \left[ \mathcal{C}_{BB'}^{vw}(x) (\mathcal{Q}^{-1})_{vw}^{bu} i \bar{\psi}_u^{B'}(x) \right] \left[ \mathcal{C}_{JC'}^{yz}(y) (\mathcal{Q}^{-1})_{yz}^{jx} i \bar{\psi}_x^{C'}(y) \right], \\
[\psi_a^A(x), \psi_e^E(y)]_+ &= \Gamma_{baf e}^{cdgh} D_{cd}^{AA'}(x) D_{gh}^{EE'}(y) \left[ \mathcal{C}_{BA'}^{jk}(x) (\mathcal{Q}^{-1})_{jk}^{bi} i \psi_i^B(x) \right] \left[ \mathcal{C}_{CE'}^{mn}(y) (\mathcal{Q}^{-1})_{mn}^{fl} i \psi_l^C(y) \right] \\
&\quad + i \mathcal{Q}_{ba}^{cd} i \mathcal{Q}_{fe}^{gh} \left[ D_{cd}^{AA'}(x), D_{gh}^{EE'}(y) \right] \left[ \mathcal{C}_{BA'}^{jk}(x) (\mathcal{Q}^{-1})_{jk}^{bi} i \psi_i^B(x) \right] \left[ \mathcal{C}_{CE'}^{mn}(y) (\mathcal{Q}^{-1})_{mn}^{fl} i \psi_l^C(y) \right], \\
[\psi_a^A(x), \bar{\psi}_b^{A'}(y)]_+ &= -i \mathcal{Q}_{ab}^{cd} D_{cd}^{AA'}(x) \delta(x-y), \\
[e_a^{AA'}(x), p_g^G(y)]_- &= \Gamma_{afhg}^{ceij} \delta_{ed} \delta^{bf} \delta_c^d D_{ij}^{GG'}(y) \left[ \delta_{bk} \delta^{lm} (\mathcal{Q}^{-1})_{lm}^{kn} i e_n^{AA'}(x) \right] \left[ \mathcal{C}_{BG'}^{yz}(y) (\mathcal{Q}^{-1})_{yz}^{hx} i \psi_x^B(y) \right], \\
[p_{AA'}^a(x), \psi_b^B(y)]_- &= \frac{i}{2} \delta_{gf} \delta^{ae} \mathcal{Q}_{eb}^{fh} \epsilon^{gcd} \psi_{Ad}(x) D_{A'hc}^B(x) \delta(x-y), \tag{122}
\end{aligned}$$

where have been used the following relations:

$$\begin{aligned}
\frac{\delta}{\delta e_a^{AA'}} &= \delta^{hi} (\mathcal{Q}^{-1})_{hi}^{ja} \left[ i p_{AA'j}(x) - \frac{1}{2} \delta_{fe} \mathcal{Q}_{jb}^{fg} \epsilon^{edc} \psi_{Ad} D_{A'gc}^B(x) \left( \mathcal{C}_{BC'}^{kl}(x) (\mathcal{Q}^{-1})_{kl}^{bm} i \bar{\psi}_m^{C'}(x) \right) \right], \\
\frac{\delta}{\delta p_{AA'}^a} &= -\delta_{ab} \delta^{cd} (\mathcal{Q}^{-1})_{cd}^{be} i e_e^{AA'}, \quad \frac{\delta}{\delta \psi_a^A} = \mathcal{C}_{AB'}^{cd} (\mathcal{Q}^{-1})_{cd}^{ab} i \bar{\psi}_b^{B'}, \quad \frac{\delta}{\delta \bar{\psi}_a^{A'}} = \mathcal{C}_{BA'}^{cd} (\mathcal{Q}^{-1})_{cd}^{ab} i \psi_b^B, \tag{123}
\end{aligned}$$

where  $\mathcal{C}_{AA'}^{ab}$  as inverse of  $D_{ab}^{AA'}$  is defined according to

$$\mathcal{C}_{AA'}^{ab} = -\frac{i\sqrt{\hbar}}{2} e^{BB'a} e_{AB'}^b n_{BA'}. \tag{124}$$

$\mathcal{C}_{AA'}^{ab}$  is the inverse of  $D_{ab}^{AA'}$  in the sense that the following relation is valid:

$$D_a^{AA'a} \mathcal{C}_{BA'b}^b = \epsilon_B^A, \tag{125}$$

where the contracted quantities  $D_a^{AA'a}$  and  $C_{AA'a}^a$  appear, which are obtained from (114) and (124),  $D_a^{AA'a} = -\frac{2i}{\sqrt{h}}n^{AA'}$ ,  $C_{AA'a}^a = -\frac{i\sqrt{h}}{2}n_{AA'}$ . In the following calculation the validity of the relation (125) can be shown,

$$\begin{aligned} D_a^{AA'a} C_{BA'b}^b &= \left[ \frac{2i}{\sqrt{h}} n^{AA'} \right] \left[ \frac{i\sqrt{h}}{2} n_{BA'} \right] = -n^{AA'} n_{BA'} = -n^\mu n^\nu \sigma_\mu^{AA'} \sigma_{\nu BA'} = -n^\mu n^\nu (\delta_{\mu\nu} \epsilon_B^A + i\epsilon_{\mu\nu\rho} \sigma_\rho^A) \\ &= -n^\mu n_\mu \epsilon_B^A = \epsilon_B^A, \quad \text{since} \quad \epsilon_{\mu\nu\rho} = -\epsilon_{\nu\mu\rho} \quad \text{and} \quad n^\mu n_\mu = -1. \end{aligned} \quad (126)$$

The relation (125) is also very important concerning the definition of the inner product defining the corresponding Hilbert space. The above constraints of classical  $\mathcal{N} = 1$  supergravity (108) lead to the following quantum constraints:

$$\hat{\mathcal{H}}_\tau |\Psi\rangle = 0, \quad \hat{\mathcal{H}}_a |\Psi\rangle = 0, \quad \hat{S}_A |\Psi\rangle = 0, \quad \hat{\hat{S}}_{A'} |\Psi\rangle = 0, \quad \hat{J}_{AB} |\Psi\rangle = 0, \quad \hat{\hat{J}}_{A'B'} |\Psi\rangle = 0. \quad (127)$$

If these quantum constraints (127) are written explicitly and the representation referring to the states expressed by  $e_a^{AA'}$  and  $\psi_a^{AA'}$ ,  $|\Psi[e, \psi]\rangle$  (118), (120), is considered, then they read as follows:

$$\begin{aligned} \hat{\mathcal{H}}_\tau |\Psi\rangle &= \left[ 16\pi G G_{abcd} \hat{\pi}^{ab} \hat{\pi}^{cd} - \frac{\sqrt{h} \hat{R}}{16\pi G} + \left( \frac{1}{2} \epsilon^{abc} \hat{\psi}_a^{A'} n_{AA'} \mathcal{D}_b \hat{\psi}_c^A + h.c. \right) \right] |\Psi\rangle \\ &= \left\{ 16\pi G G_{abcd} \frac{e^{AA'(a} \left[ -i\delta_{hg} \delta^{b)i} \mathcal{Q}_{ei}^{fh} \delta_f^g \frac{\delta}{\delta e_e^{AA'}} - \frac{i}{2} \delta_{ji} \delta^{b)h} \mathcal{Q}_{he}^{ik} \epsilon^{jgf} \psi_{Ag} D_{A'kf}^B \frac{\delta}{\delta \psi_e^B} \right]}{2} \right. \\ &\quad \cdot \frac{e^{BB'(c} \left[ -i\delta_{hg} \delta^{d)i} \mathcal{Q}_{ei}^{fh} \delta_f^g \frac{\delta}{\delta e_e^{AA'}} - \frac{i}{2} \delta_{ji} \delta^{d)h} \mathcal{Q}_{he}^{ik} \epsilon^{jgf} \psi_{Ag} D_{A'kf}^B \frac{\delta}{\delta \psi_e^B} \right]}{2} \\ &\quad \left. - \frac{\sqrt{h} R}{16\pi G} + \left[ \frac{1}{2} \epsilon^{abc} \left( -i \mathcal{Q}_{fa}^{gh} D_{gh}^{A'B} \frac{\delta}{\delta \psi_f^B} \right) n_{AA'} \mathcal{D}_b \psi_c^A + h.c. \right] \right\} |\Psi(e, \psi)\rangle = 0, \end{aligned} \quad (128)$$

$$\begin{aligned} \hat{\mathcal{H}}_a |\Psi\rangle &= \left[ -2\hat{h}_{ab} \mathcal{D}_c \hat{\pi}^{bc} + \left( \frac{1}{2} \epsilon^{bcd} \hat{\psi}_b^{A'} \hat{e}_{AA'a} \mathcal{D}_c \hat{\psi}_d^A + h.c. \right) \right] |\Psi\rangle \\ &= \left\{ -2\hat{h}_{ab} \mathcal{D}_c \frac{\hat{e}^{AA'(b} \left[ -i\delta_{gf} \delta^{c)h} \mathcal{Q}_{dh}^{eg} \delta_e^f \frac{\delta}{\delta e_d^{AA'}} - \frac{i}{2} \delta_{ih} \delta^{c)g} \mathcal{Q}_{gd}^{hj} \epsilon^{ife} \psi_{Af} D_{A'je}^B \frac{\delta}{\delta \psi_d^B} \right]}{2} \right. \\ &\quad \left. + \left[ \frac{1}{2} \epsilon^{bcd} \left( -i \mathcal{Q}_{fb}^{gh} D_{gh}^{A'B} \frac{\delta}{\delta \psi_f^B} \right) \hat{e}_{AA'a} \mathcal{D}_c \hat{\psi}_d^A + h.c. \right] \right\} |\Psi(e, \psi)\rangle = 0, \end{aligned} \quad (129)$$

$$\begin{aligned} \hat{S}_A |\Psi\rangle &= \left[ \epsilon^{abc} \hat{e}_{AA'a} \mathcal{D}_b \hat{\psi}_c^{A'} - i4\pi G \hat{p}_{AA'}^a \hat{\psi}_a^{A'} \right] |\Psi\rangle \\ &= \left[ \epsilon^{abc} e_{AA'a} \mathcal{D}_b \left( -i \mathcal{Q}_{fc}^{gh} D_{gh}^{A'B} \frac{\delta}{\delta \psi_f^B} \right) - i4\pi G \left( -i\delta_{ed} \delta^{af} \mathcal{Q}_{bf}^{ce} \delta_c^d \frac{\delta}{\delta e_b^{AA'}} \right. \right. \\ &\quad \left. \left. - \frac{i}{2} \delta_{gf} \delta^{ae} \mathcal{Q}_{eb}^{fh} \epsilon^{gdc} \psi_{Ad} D_{A'hc}^B \frac{\delta}{\delta \psi_b^B} \right) \cdot \left( -i \mathcal{Q}_{fa}^{gh} D_{gh}^{A'B} \frac{\delta}{\delta \psi_f^B} \right) \right] |\Psi(e, \psi)\rangle = 0, \end{aligned} \quad (130)$$

$$\begin{aligned} \hat{\hat{S}}_{A'} |\Psi\rangle &= \left[ \epsilon^{abc} \hat{e}_{AA'a} \mathcal{D}_b \hat{\psi}_c^A + i4\pi G \hat{\psi}_a^{A'} \hat{p}_{AA'}^a \right] |\Psi\rangle \\ &= \left[ \epsilon^{abc} e_{AA'a} \mathcal{D}_b \psi_c^A + i4\pi G \psi_a^{A'} \left( -i\delta_{ed} \delta^{af} \mathcal{Q}_{bf}^{ce} \delta_c^d \frac{\delta}{\delta e_b^{AA'}} \right. \right. \\ &\quad \left. \left. - \frac{i}{2} \delta_{gf} \delta^{ae} \mathcal{Q}_{eb}^{fh} \epsilon^{gdc} \psi_{Ad} D_{A'hc}^B \frac{\delta}{\delta \psi_b^B} \right) \right] |\Psi(e, \psi)\rangle = 0, \end{aligned} \quad (131)$$

$$\begin{aligned}
\hat{J}_{AB}|\Psi\rangle &= \left[ \hat{e}_{(A}^{A'a} \hat{p}_{B)A'a} - \frac{1}{2} \hat{\psi}_{(A}^a \epsilon_{abc} \hat{\psi}^{A'b} \hat{e}_{B)A'}^c \right] |\Psi\rangle \\
&= \left[ e_{(A}^{A'a} \left( -i\delta_{ed} \mathcal{Q}_{ba}^{ce} \delta_c^d \frac{\delta}{\delta e_b^{A'}} - \frac{i}{2} \delta_{fe} \mathcal{Q}_{ab}^{eg} \epsilon^{f dc} \psi_{B)d} D_{A'gc}^C \frac{\delta}{\delta \psi_b^C} \right) \right. \\
&\quad \left. - \frac{1}{2} \psi_{(A}^a \epsilon_{abc} \left( -i\delta^{be} \mathcal{Q}_{fe}^{gh} D_{cd}^{A'C} \frac{\delta}{\delta \psi_f^C} \right) e_{B)A'}^c \right] |\Psi(e, \psi)\rangle = 0,
\end{aligned} \tag{132}$$

$$\begin{aligned}
\hat{\bar{J}}_{A'B'}|\Psi\rangle &= \left[ \hat{e}_{(A'}^{Aa} \hat{p}_{B')Aa} - \frac{1}{2} \hat{\bar{\psi}}_{(A'}^a \epsilon_{abc} \hat{\bar{\psi}}^{Ab} \hat{e}_{B')A}^c \right] |\Psi\rangle \\
&= \left[ e_{(A'}^{Aa} \left( -i\delta_{ed} \mathcal{Q}_{ba}^{ce} \delta_c^d \frac{\delta}{\delta e_b^{B'}} - \frac{i}{2} \delta_{fe} \mathcal{Q}_{ab}^{eg} \epsilon^{f dc} \psi_{B')d} D_{A'gc}^C \frac{\delta}{\delta \bar{\psi}_b^C} \right) \right. \\
&\quad \left. - \frac{1}{2} \epsilon_{(A'D'} \left( -i\delta^{ae} \mathcal{Q}_{fe}^{gh} D_{gh}^{D'C} \frac{\delta}{\delta \bar{\psi}_f^C} \right) \epsilon_{abc} \psi^{Ab} e_{B')A}^c \right] |\Psi(e, \psi)\rangle = 0.
\end{aligned} \tag{133}$$

To define the Hilbert space of the states,  $|\Psi[e, \psi]\rangle$  or  $|\Psi[p, \bar{\psi}]\rangle$  respectively, of the quantum theory of  $\mathcal{N} = 1$  supergravity based on the quaternionic quantization principle,  $\mathcal{H}_{\mathcal{N}=1}$ , an inner product has to be specified finally. The inner product  $\langle \cdot | \cdot \rangle$  between two quantum states of quantum supergravity under presupposition of the quaternionic quantization principle is formulated as follows:

$$\langle \Phi | \Psi \rangle = \int \mathcal{D}e \mathcal{D}\psi \mathcal{D}\bar{\psi} \bar{\Phi}(e, \bar{\psi}) \Psi(e, \psi) \exp \left[ - \int d^3x (\mathcal{Q}^{-1})_{cd}^{ab} i C_{AA'}^{cd}(x) \psi_a^A(x) \bar{\psi}_b^{A'}(x) \right] D^{-1}(e), \tag{134}$$

where  $D(e)$  is defined as

$$D(e) = \prod_x \det [-i C_{AA'}^{ab}(x)]. \tag{135}$$

The definition of the inner product according to (134) represents the corresponding generalization of the inner product given in [102] to the case of the quaternionic quantization principle. Besides incorporating  $(\mathcal{Q}^{-1})_{cd}^{ab}$  to the inner product, the quantity  $C_{AA'}^{ab} = -\epsilon^{abc} e_{AA'c}(x)$  appearing in the usual formulation of the inner product of quantum supergravity has been replaced by  $\mathcal{C}_{AA'}^{ab}$ , which has already been defined in (124). The definition of the inner product according to (134) maintains that the operators  $\hat{\psi}_a^A$  and  $\hat{\bar{\psi}}_a^{A'}$  are still hermitian conjugated quantities to each other in this generalized quantization scenario,

$$\langle \Phi | \hat{\bar{\psi}}_a^{A'} | \Psi \rangle = \langle \hat{\psi}_a^A \Phi | \Psi \rangle. \tag{136}$$

This can be shown as follows:

$$\begin{aligned}
\langle \Phi | \hat{\bar{\psi}}_a^{A'} | \Psi \rangle &= \int \mathcal{D}e \mathcal{D}\psi \mathcal{D}\bar{\psi} \bar{\Phi}(e, \bar{\psi}) \left[ -i \mathcal{Q}_{ba}^{cd} D_{cd}^{AA'} \frac{\delta \Psi(e, \psi)}{\delta \psi_b^A} \right] \exp \left[ - \int d^3x (\mathcal{Q}^{-1})_{gh}^{ef} i \mathcal{C}_{BB'}^{cd}(x) \psi_e^B(x) \bar{\psi}_f^{B'}(x) \right] D^{-1}(e), \\
&= \int \mathcal{D}e \mathcal{D}\psi \mathcal{D}\bar{\psi} i \mathcal{Q}_{ba}^{cd} D_{cd}^{AA'} \frac{\delta}{\delta \psi_b^A} \left[ \Phi(e, \bar{\psi}) \exp \left[ - \int d^3x (\mathcal{Q}^{-1})_{gh}^{ef} i \mathcal{C}_{BB'}^{cd}(x) \psi_e^B(x) \bar{\psi}_f^{B'}(x) \right] D^{-1}(e) \right] \Psi(e, \psi), \\
&= \int \mathcal{D}e \mathcal{D}\psi \mathcal{D}\bar{\psi} i \mathcal{Q}_{ba}^{cd} D_{cd}^{AA'} \frac{\delta}{\delta \psi_b^A} \left[ - \int d^3x (\mathcal{Q}^{-1})_{gh}^{ef} i \mathcal{C}_{BB'}^{cd}(x) \psi_e^B(x) \bar{\psi}_f^{B'}(x) \right] \\
&\quad \times \Phi(e, \bar{\psi}) \exp \left[ - \int d^3x (\mathcal{Q}^{-1})_{gh}^{ef} i \mathcal{C}_{BB'}^{cd}(x) \psi_e^B(x) \bar{\psi}_f^{B'}(x) \right] D^{-1}(e) \Psi(e, \psi), \\
&= \int \mathcal{D}e \mathcal{D}\psi \mathcal{D}\bar{\psi} \bar{\Phi}(e, \bar{\psi}) \bar{\psi}_a^{A'} \exp \left[ - \int d^3x (\mathcal{Q}^{-1})_{gh}^{ef} i \mathcal{C}_{BB'}^{cd}(x) \psi_e^B(x) \bar{\psi}_f^{B'}(x) \right] D^{-1}(e), \\
&= \int \mathcal{D}e \mathcal{D}\psi \mathcal{D}\bar{\psi} \hat{\bar{\psi}}_a^{A'} \bar{\Phi}(e, \bar{\psi}) \exp \left[ - \int d^3x (\mathcal{Q}^{-1})_{gh}^{ef} i \mathcal{C}_{BB'}^{cd}(x) \psi_e^B(x) \bar{\psi}_f^{B'}(x) \right] D^{-1}(e) = \langle \hat{\psi}_a^A \Phi | \Psi \rangle,
\end{aligned} \tag{137}$$

where has been used partial integration in the second step including the boundary condition

$$- \int \mathcal{D}e \mathcal{D}\bar{\psi} i\mathcal{Q}_{ba}^{cd} D_{cd}^{AA'} \left[ \Phi(e, \bar{\psi}) \exp \left[ - \int d^3x (\mathcal{Q}^{-1})_{gh}^{ef} i\mathcal{C}_{BB'}^{cd}(x) \psi_e^B(x) \bar{\psi}_f^{B'}(x) \right] D^{-1}(e) \Psi(e, \bar{\psi}) \right] \Big|_{-\infty}^{\infty} = 0, \quad (138)$$

and the representation of the operator  $\hat{\psi}_a^A$  with respect to the states depending on  $e$  and  $\bar{\psi}$ ,  $|\Psi(e, \bar{\psi})\rangle$ , which has been given in (120). In the fourth step has been performed the following relation:

$$i\mathcal{Q}_{ba}^{cd} D_{cd}^{AA'} \frac{\delta}{\delta \bar{\psi}_b^{A'}(x)} \left[ - \int d^3x' (\mathcal{Q}^{-1})_{gh}^{ef} i\mathcal{C}_{BB'}^{gh}(x') \psi_e^B(x') \bar{\psi}_f^{B'}(x') \right] = \psi_a^A(x) \left[ - \int d^3x' (\mathcal{Q}^{-1})_{gh}^{ef} i\mathcal{C}_{BB'}^{gh}(x') \psi_e^B(x') \bar{\psi}_f^{B'}(x') \right], \quad (139)$$

which is based on the calculation

$$-i\mathcal{Q}_{ba}^{cd} D_{cd}^{AA'} (\mathcal{Q}^{-1})_{gh}^{eb} i\mathcal{C}_{BA'}^{gh} \psi_e^B = - \left[ i \frac{\delta^{cd}}{3} \mathcal{P}_{ab} \right] \left[ 3\delta_{gh} (\mathcal{P}^{-1})^{eb} i \right] D_{cd}^{AA'} \mathcal{C}_{BA'}^{gh} \psi_e^B = \delta_a^e D_c^{AA'} \mathcal{C}_{BA'}^g \psi_e^B = \delta_a^e \epsilon_B^A \psi_e^B = \psi_a^A, \quad (140)$$

where has been used that  $\mathcal{P}^{ab} (\mathcal{P}^{-1})_{bc} = \delta_c^a$  as well as the relation (125), whose validity has been shown in (126). Also the Fourier transformations between the representation  $|\Psi(e, \bar{\psi})\rangle$  and  $|\tilde{\Psi}(e, \bar{\psi})\rangle$ , which are wave-functionals, is defined by the inner product (134), and the Fourier transformation corresponding to (134) reads as follows:

$$\tilde{\Psi}(e, \bar{\psi}) = D^{-1}(e) \int \mathcal{D}\psi \Psi(e, \bar{\psi}) \exp \left[ - \int d^3x (\mathcal{Q}^{-1})_{cd}^{ab} i\mathcal{C}_{AA'}^{cd}(x) \psi_a^A(x) \bar{\psi}_b^{A'}(x) \right]. \quad (141)$$

## VII. SUMMARY AND DISCUSSION

In this paper has been presented a generalized quantization principle for quantum theory based on an algebra containing quaternions, which has been applied to quantum mechanics as well as to canonical quantum general relativity and to  $\mathcal{N}=1$  canonical quantum supergravity. The presented theory can be seen as an alternative approach to formulate general quantum theory and especially to quantize general relativity. The quaternionic quantization principle assumes that the components of a variable belonging to a special space-time direction do not only fulfil nontrivial commutation relations with the corresponding component of the canonical conjugated variable, but also with the other components of this variable. The additional commutation relations are assumed to be proportional to another direction than the usual complex direction in the space of quaternions. This implies also additional commutation relations between the several components of the special variable with each other, leading to noncommutative geometry in case of quantum mechanics and leading to commutation relations between the components of the gravitational field in quantum gravity. This has its origin in the fact that the components of quaternions do not commute with each other and accordingly also the components of the generalized quantization tensor, which is contained in the generalized expressions of the operators, do not commute with each other.

Especially the application of this quantization principle to supergravity contains special intricacies, which are related to the necessity to use Dirac brackets instead of Poisson brackets to obtain the corresponding quantum theory from the classical theory. Accordingly the quaternionic quantization principle has to be adapted to the special conditions of supergravity, what has been done for the special case of  $\mathcal{N} = 1$  supergravity in this paper. Because of the appearance of Dirac brackets in case of supergravity additional commutation relations to the usual commutation relations based on the Heisenberg algebra have to be considered, which are defined by the commutation relations between the several operators and are decisively extended in case of a presupposition of the quaternionic quantization principle. Besides, the inner product of canonical quantum supergravity has to be modified to maintain that the conjugated quantity to the Rarita Schwinger field is still the hermitian adjoint quantity.

The relation between this generalization of the quantization principle of general quantum theory to the usual one is determined by the new dimensionless parameter  $\varkappa$ . If  $\varkappa$  goes to zero, the generalized theory is approximatively equal to the usual quantum theory. In principle, it is also thinkable that the quaternionic quantization principle is just valid for the special case of the quantization of gravity, as quantization of usual general relativity or supergravity respectively. In this case quantum mechanics and quantum field theory would not have to be changed, but quantum gravity would be based on such a generalized quantization principle.

The decisive intention of the paper consists in the assumption that the formulation of a quantum theory of general relativity could not only presuppose a generalization of the classical approximation or at least its formulation, but could alternatively or additionally presuppose a generalization of the quantization principle as way to obtain a quantum theory from a classical theory. The presented idea of such a generalization of the concept of quantization can be interpreted as a special manifestation of this principle consideration, which is based on a suggesting generalization, because there is no reason why quantum theory should be restricted to complex Hilbert spaces and why the commutation relations between the components of the variables and the canonical conjugated variables belonging to different space-time directions should be assumed to commute. This means that if the possibility of a generalization of quantum theory as it is already postulated in noncommutative geometry with respect to the several components of space-time coordinates is assumed to make sense, the presented theory appears as a very promising candidate for such a generalization, which can in principle also be transferred to other formulations of general relativity as the loop representation or to extended supergravity theories.

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